# Free fall periodic orbits of three bodies: new remarkable properties

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**Abstract**

This article describes a discovery of new remarkable properties in the free fall periodic orbits of equal masses [1] for the Newtonian three body problem. First, we figured out these new properties merely by observing simulations of the 30 periodic free fall orbits reported in [1] with the advanced software for integration of ODEs called the Taylor Center [4]. Then we verified those 30 properties numerically utilizing the same software, whose high accuracy was very instrumental for this purpose.

Keywords: Three-body problem, Periodic, Free fall, Taylor integration

## 1 Introduction

The three body motion under the Newtonian gravitation has been intensively studied since Isaac Newton over 300 years, still presenting challenges. Here we are speaking about the plane case

$$\begin{aligned}
\ddot{x}_1 &= m_3(x_3 - x_1)r_{31} - m_2(x_1 - x_2)r_{12} \\
\ddot{y}_1 &= m_3(y_3 - y_1)r_{31} - m_2(y_1 - y_2)r_{12} \\
\ddot{x}_2 &= m_1(x_1 - x_2)r_{12} - m_3(x_2 - x_3)r_{23} \\
\ddot{y}_2 &= m_1(y_1 - y_2)r_{12} - m_3(y_2 - y_3)r_{23} \\
\ddot{x}_3 &= m_2(x_2 - x_3)r_{23} - m_1(x_3 - x_1)r_{31} \\
\ddot{y}_3 &= m_2(y_2 - y_3)r_{23} - m_1(y_3 - y_1)r_{31}
\end{aligned} \tag{1}$$

at the initial positions $\mathbf{q}_1 = (x_1,\ y_1),\ \mathbf{q}_2 = (x_2,\ y_2),\ \mathbf{q}_3 = (x_3,\ y_3)$, where

$$\begin{aligned}
r_{12} &= ((x_1 - x_2)^2 + (y_1 - y_2)^2)^{-3/2} \\
r_{23} &= ((x_2 - x_3)^2 + (y_2 - y_3)^2)^{-3/2} \\
r_{31} &= ((x_3 - x_1)^2 + (y_3 - y_1)^2)^{-3/2}.
\end{aligned}$$

Besides the special cases of elliptic (and other conics) by Euler and Lagrange, no other versions of regular motion of three bodies were known for a long time. Since emergence of computers, also numeric simulations were used to explore the orbits for various initial settings, most of which generated a chaotic motion.

So more surprising were discoveries of remarkable types of plane periodic motion of three bodies obtained in computer assisted researches. Such were the discovery of ...

- Choreography, when three bodies move along the same periodic curve one after the other [6];
- Periodic and relatively periodic orbits [7];
- Free fall periodic orbits [8] meaning that three bodies have zero velocity at the initial moment - the topic of this paper.

In 2018 Xiaoming Li and Shijun Liao [1-3] discovered hundreds of settings for initially resting three bodies making periodic motion: i.e. the bodies started at the given rest points, and returned back to their initial rest points after motion along sophisticated orbits during a period $T$.

We call the moments of time when all three bodies rest *the break points* paying attention to the triangular formation at the moments of rest.

It's worth particular mentioning that Xiaoming Li and Shijun Liao [2, 3] developed and ran their search of various initial settings of the rest points with the only goal to figure out solutions having periods. They did it by watching for the values of the target function

$$\sum_{i=1}^{3} \left( |\mathbf{q}_i(t) - \mathbf{q}_i(0)| + |\dot{\mathbf{q}}_i(t) - \dot{\mathbf{q}}_i(0)| \right), \qquad \dot{\mathbf{q}}_i(0) = 0 \tag{2}$$

whether its values are close to zero (with the given threshold).

However, despite setting the goal of obtaining merely periodic solutions ending in the same initial points, all their 30 solutions for equal masses demonstrated some additional properties *not specified* in the search criterion:

1. As pointed out by the authors [3], in all those discovered periodic trajectories, the initial rest formations at the moments $kT \quad (k = 0,\ 1,\ ...)$ happened to be not the only one. In every periodic lap of the trajectory the number of break points was exactly 2: namely the initial moment and the moment $T/2$ where all three bodies come to rest, but at a formation differing from the initial one. That meant that ...
2. The three bodies oscillated between two formations: the initial and second set of the rest points (specific for each simulation).

Though the authors did mention it [3], they did not provide any explanation for this unexpected *"side effect"* taking place in all 30 cases.

We discovered other even more remarkable *side effects* not specified as a search criterion too. They appear as certain *exact relations,* though not in all, but only in a few of the 30 simulations. Emergence of such uninvited properties in a massive search is puzzling. These properties are reported in the section "Newly discovered properties".

## 2 A proof for item 1

The fact that the properties 1, 2 took place even though never specified as a search criterion has an explanation, which follows from the next Theorem.

**Theorem 1** *Free fall periodic orbits have exactly two sets (two formations) of rest points so that the bodies oscillate between them.*

**Proof 1** *(Provided by Richard Montgomery in a private correspondence). We may assume that the initial rest formation happens at* $t = 0$ *so that* $\dot{\mathbf{q}}(0) = 0$*. By time reversal invariance we have*

$$\mathbf{q}_i(-t) = \mathbf{q}_i(t), \quad i = 1,\ 2,\ 3 \tag{3}$$

*for all* $t$*. Let* $T > 0$ *be the period, i.e. the smallest time such that*

$$\mathbf{q}_i(T + t) = \mathbf{q}_i(t), \quad i = 1,\ 2,\ 3 \tag{4}$$

*for all* $t$*. (Here* $T > 0$ *since* $t = 0$ *is not an equilibrium point where the orbit degenerated into a point). Together then, we get* $\mathbf{q}_i(-T/2) = \mathbf{q}_i(-T/2 + T) = \mathbf{q}_i(T/2)$*. Let's show that* $\mathbf{q}_i(T/2)$ *is a second formation of the rest points along the orbit, i.e. that* $\dot{\mathbf{q}}_i(T/2) = 0$*. Note that* $\mathbf{q}_i(-T/2 - h) = \mathbf{q}_i(T/2 + h)$ *because of symmetry (3), while* $\mathbf{q}_i(-T/2 - h) = \mathbf{q}_i(T/2 - h)$ *because of periodicity (4). Therefore,* $\mathbf{q}_i(T/2 + h) = \mathbf{q}_i(T/2 - h)$ *for all* $h$*. Differentiating it with respect to* $h$ *we get* $\dot{\mathbf{q}}_i(T/2 + h) = -\dot{\mathbf{q}}_i(T/2 - h)$*, or* $\dot{\mathbf{q}}_i(T/2 + h) + \dot{\mathbf{q}}_i(T/2 - h) = 0$*. Then obtaining the* $\lim\limits_{h \to 0}$ *we come to* $\dot{\mathbf{q}}_i(T/2) = 0$ *meaning that* $T/2$ *is also a break point of some rest formation.*

*Now let's prove that the rest formation at the break point* $t = T/2$ *is distinct from that at* $t = 0$*. Suppose the opposite, i.e. that formation* $\mathbf{q}_i(T/2) = \mathbf{q}_i(0) = q_{i0},\ \ i = 1,\ 2,\ 3$*. As we know that also* $\dot{\mathbf{q}}_i(T/2) = \dot{\mathbf{q}}_i(0) = 0$*, we have two initial value problems for ODEs (1) with the same initial values though for different moment of time:*

$$\begin{aligned} \mathbf{q}_i(T/2) &= q_{i0}, \quad \dot{\mathbf{q}}_i(T/2) = 0 \\ \mathbf{q}_i(0) &= q_{i0}, \quad \dot{\mathbf{q}}_i(0) = 0. \end{aligned}$$

*These IVPs are for the same autonomous Newtonian ODEs (1) not depending on* $t$*, therefore these solutions are identical, i.e.* $\mathbf{q}_i(t) = \mathbf{q}_i(t + T/2)$ *meaning* $T/2$*periodicity - which is impossible. Therefore, the formations* $\mathbf{q}_i(T/2)$ *and* $\mathbf{q}_i(0)$ *are not equal, thus we proved that there are at least two distinct rest formations at two break points.*

*Let's prove that there can be only exactly two break points. Suppose that there exist a third break point $t_1$, $\quad 0 < t_1 < T/2$. That means that the path would have to stop at $t_1$ and then reverse, returning to 0 thus never making it to $T/2$, contradicting that $T$ is the period, and $T/2$ - a half-period. Therefore, the orbits in free fall of three bodies have a period $T$, they oscillate between exactly two sets of rest formations.*

This Theorem explains why the search process of periodic orbits starting with a break point delivered the orbits all having the second break points.

All such orbits are cataloged and displayed as movies at the authors' site [1]. The first 30 simulations in the table present the cases of all three masses equal 1. These simulations also come with the free installation of the software Taylor Center (where you can watch them in real time with the resolution much higher than in the movies [1]).

In the next section we report more new properties, taking place, however, not in all 30 of the above mentioned orbits. Those properties were discovered by chance in numeric experiments with the 30 simulations performed and verified with the Taylor Center software.

## 3 The new properties.

Here we summarize the newly discovered properties taking place in 12 of the 30 orbits.

The Table 1 below shows in which of the 30 simulations [1] these earlier unknown properties take place.

We consider two triangular formations of the 3 bodies: the initial $\triangle ABC$ at the moment $t = 0$ and the second $\triangle A'B'C'$ at the moment $t = T/2$, where $\triangle ABC$ and $\triangle A'B'C'$ are congruent with or without reflection. This means the equality of the corresponding angles $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.

Let the bodies #1, #2, #3 at the initial moment reside correspondingly at the vertices $A$, $B$, $C$ of the $\triangle ABC$. Their trajectories, however, may not necessarily lead to the corresponding vertices $A'$, $B'$, $C'$ of the $\triangle A'B'C'$, as some simulations below demonstrate. Among the data collected by the research program, there are the permutations $(\alpha\beta\gamma)$, where the identity permutation is denoted $Id = (123)$. If the trajectories of the bodies #1, #2, #3 (black, red, and blue) lead from the vertices $A$, $B$, $C$ to the corresponding vertices $A'$, $B'$, $C'$ (no matter whether $\triangle A'B'C'$ is a reflection of $\triangle ABC$), the corresponding permutation is $Id$; otherwise the permutation differs from $Id$.

These newly discovered properties in 12 of the 30 orbits are the following:

3. In the moments $\frac{1}{4}T$ and $\frac{3}{4}T$ the bodies are either in syzygy (item 5), or they form an isosceles triangle (item 6).

4. The triangle formation at the second break point $\frac{1}{2}T$ is *congruent* to the initial triangle.

5. In 3 of the 12 orbits in the moments $\frac{1}{4}T$ and $\frac{3}{4}T$ the second triangle is a result of 180° rotation of the initial triangle so that both triangles and respective parts of orbits are *symmetric* over the *central point* lying on the syzygy, one of the bodies being in the middle. At that, the three vectors of the velocities in the moments $\frac{1}{4}T$ and $\frac{3}{4}T$ are reciprocally parallel. However...

6. In the remaining 9 orbits the edges are not parallel, and both triangles are in the relation of reflection, i.e. the two respective parts of orbits are symmetric over some line of symmetry, which, however, is not necessarily the line of syzygy. Specifically...

7. If there is no permutation (i.e. $Id$ takes place), then in the moments $\frac{1}{4}T$ and $\frac{3}{4}T$ the 3 bodies are in syzygy on the line of symmetry, otherwise in the moments $\frac{1}{4}T$ and $\frac{3}{4}T$ the bodies are not in syzygy, forming an isosceles triangle.

The properties (3-7) are not mentioned in the original sources [1-3], and therefore they are new. Unlike the properties (1-2) proven to take place in any free fall periodic orbit, at the moment it's not known the conditions leading to the 12 cases of congruency discussed here.

| | At $t = T/2$ | | | | At $t = T/4$ | |
|---|---|---|---|---|---|---|
| # | Congruency | Parallel edges | Symmetry | Perm | Isosceles | Syzygy |
| 1 | | | | | | |
| 2 | | | | | | |
| 3 | | | | | | |
| 4 | Yes | | Reflection | (321) | Yes | |
| 5 | | | | | | |
| 6 | Yes | | Reflection | (321) | Yes | |
| 7 | | | | | | |
| 8 | Yes | | Reflection | $Id$ | | Yes |
| 9 | | | | | | |
| 10 | Yes | | Reflection | $Id$ | | Yes |
| 11 | | | | | | |
| 12 | | | | | | |
| 13 | | | | | | |
| 14 | Yes | Yes | Central | (132) | Yes | Yes |
| 15 | Yes | Yes | Central | (132) | Yes | Yes |
| 16 | | | | | | |
| 17 | | | | | | |
| 18 | Yes | | Reflection | $Id$ | | Yes |
| 19 | Yes | | Reflection | (213) | Yes | |
| 20 | | | | | | |
| 21 | | | | | | |
| 22 | Yes | | Reflection | $Id$ | | Yes |
| 23 | | | | | | |
| 24 | | | | | | |
| 25 | Yes | | Reflection | $Id$ | | Yes |
| 26 | | | | | | |
| 27 | Yes | Yes | Central | (213) | Yes | Yes |
| 28 | | | | | | |
| 29 | Yes | | Reflection | $Id$ | | Yes |
| 30 | | | | | | |

Tab. 1: New properties in the 30 free fall cases.

## 4 How the triplets of initial points were obtained

As explained in [1-3], the authors fixed the points $\mathbf{q}_1 = (-0.5,\ 0)$, $\mathbf{q}_2 = (0.5,\ 0)$, while the goal of the search algorithm was to obtain points $\mathbf{q}_3$ such that the target function (2) be near zero with the specified accuracy. Below is a scattered graph for the 30 points $\mathbf{q}_3$ obtained in the search process [1-3]:

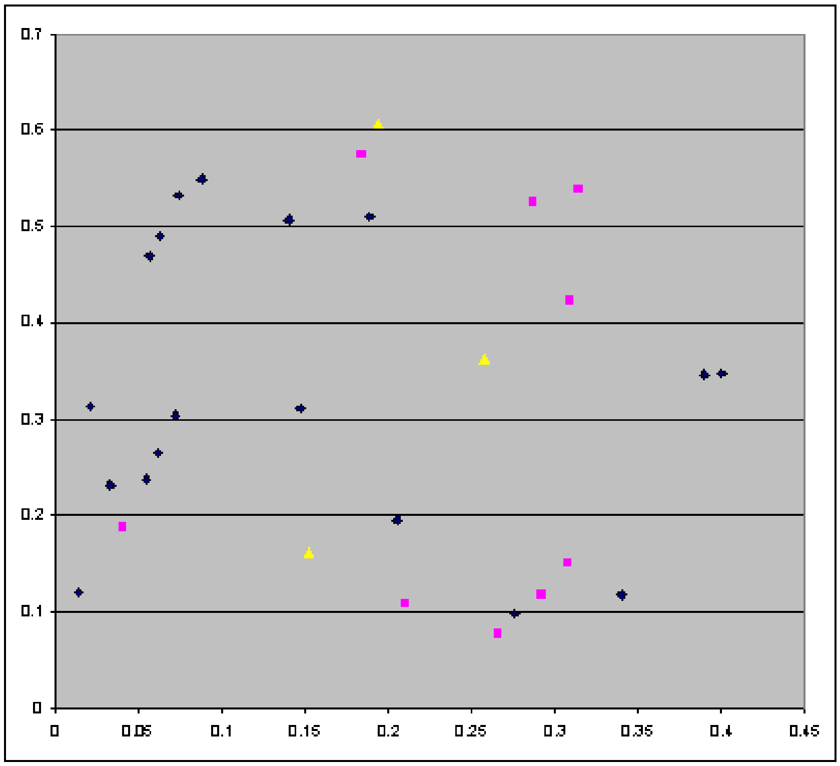



Figure 1. The 30 values for the third initial point $\mathbf{q}_3$ obtained in a search algorithm

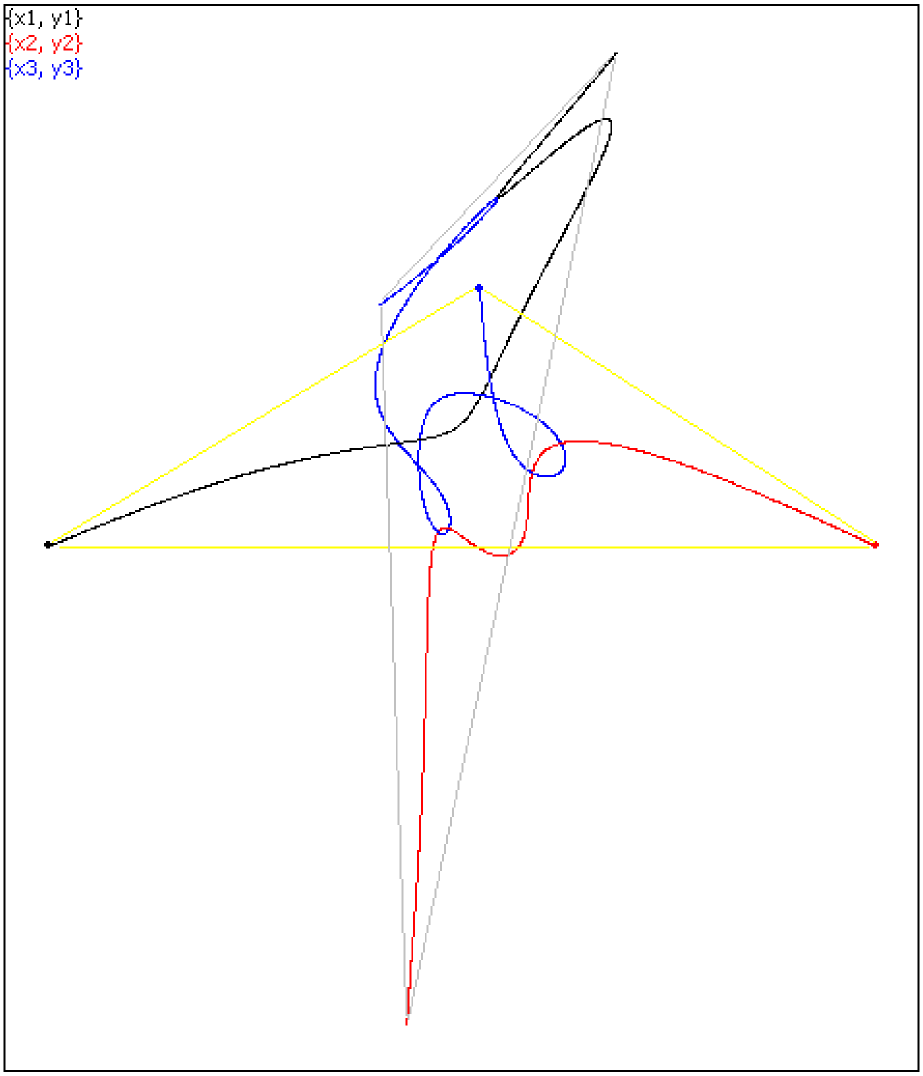
The three yellow points correspond to the cases of the central symmetry, the 9 magenta points correspond to the cases of reflection, and the remaining blue points correspond to the orbits with no special properties. This graph does not reveal any remarkable pattern. There is no mentioning in [1-3] whether the search algorithm delivered all existing points $\mathbf{q}_3$ in the given bounded area of search, and whether the number of points $\mathbf{q}_3$ is finite or infinite.

# 5 Some illustrations

Below are a few illustrations of the newly discovered properties in some of the 12 cases of periodic free fall (the Table 1 summarizes which of the original 30 simulations possess the newly discovered properties.) Illustrations below present the trajectories of the three bodies (in black, red, and blue), and the triangle formations at the moments $t = 0$ (yellow) and $t = T/2$ (gray). The high resolution images of all the 12 orbits with the special properties may be found here [9].

All the 30 simulations may be also viewed dynamically as real time animation within the Taylor center software, as explained in the Appendix.

In most cases of the 30 simulations the second formation is typically a triangle dissimilar to the initial one, say like in the Simulation 1.

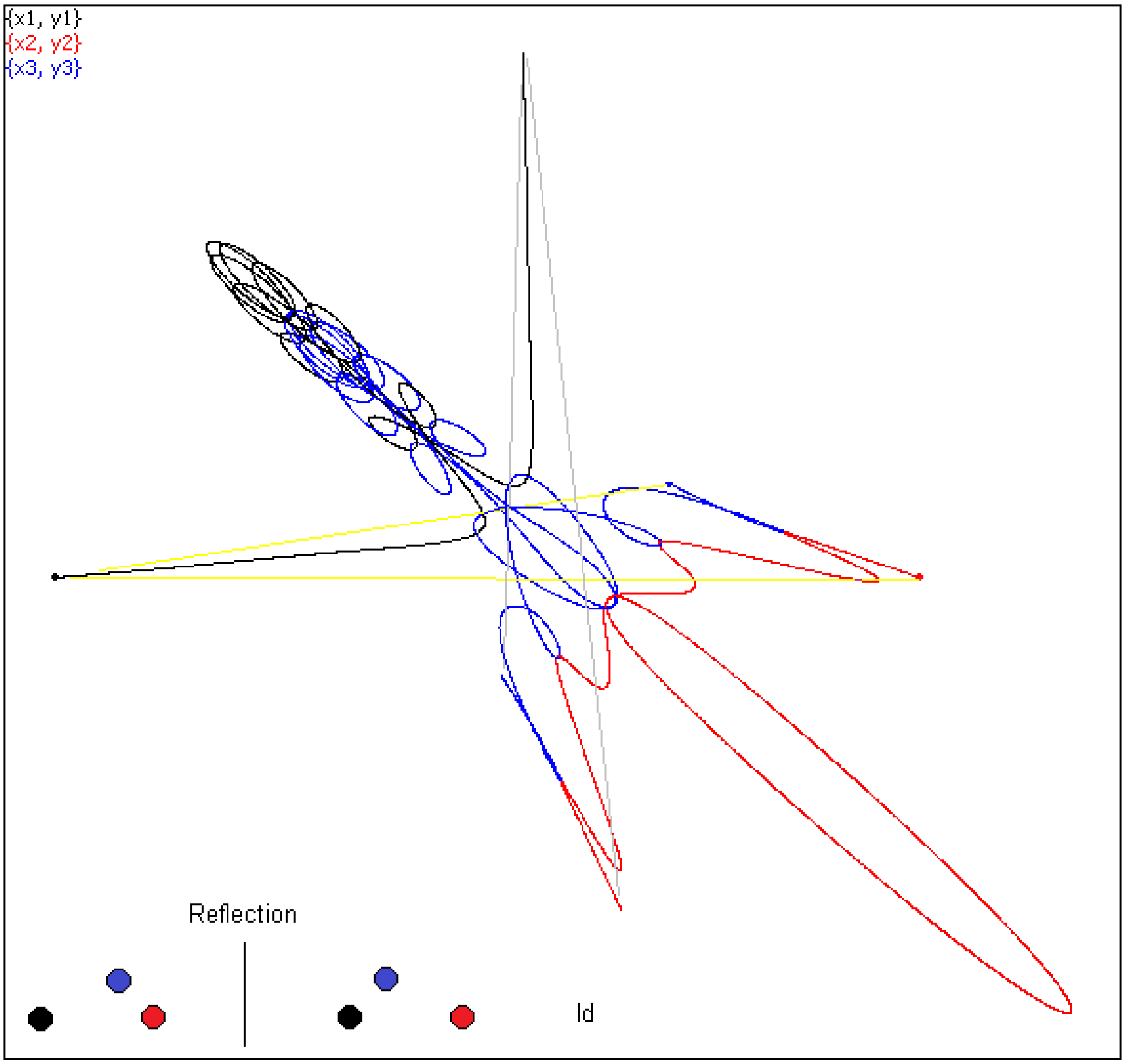



Simulation #1: the second triangle and the first are dissimilar.

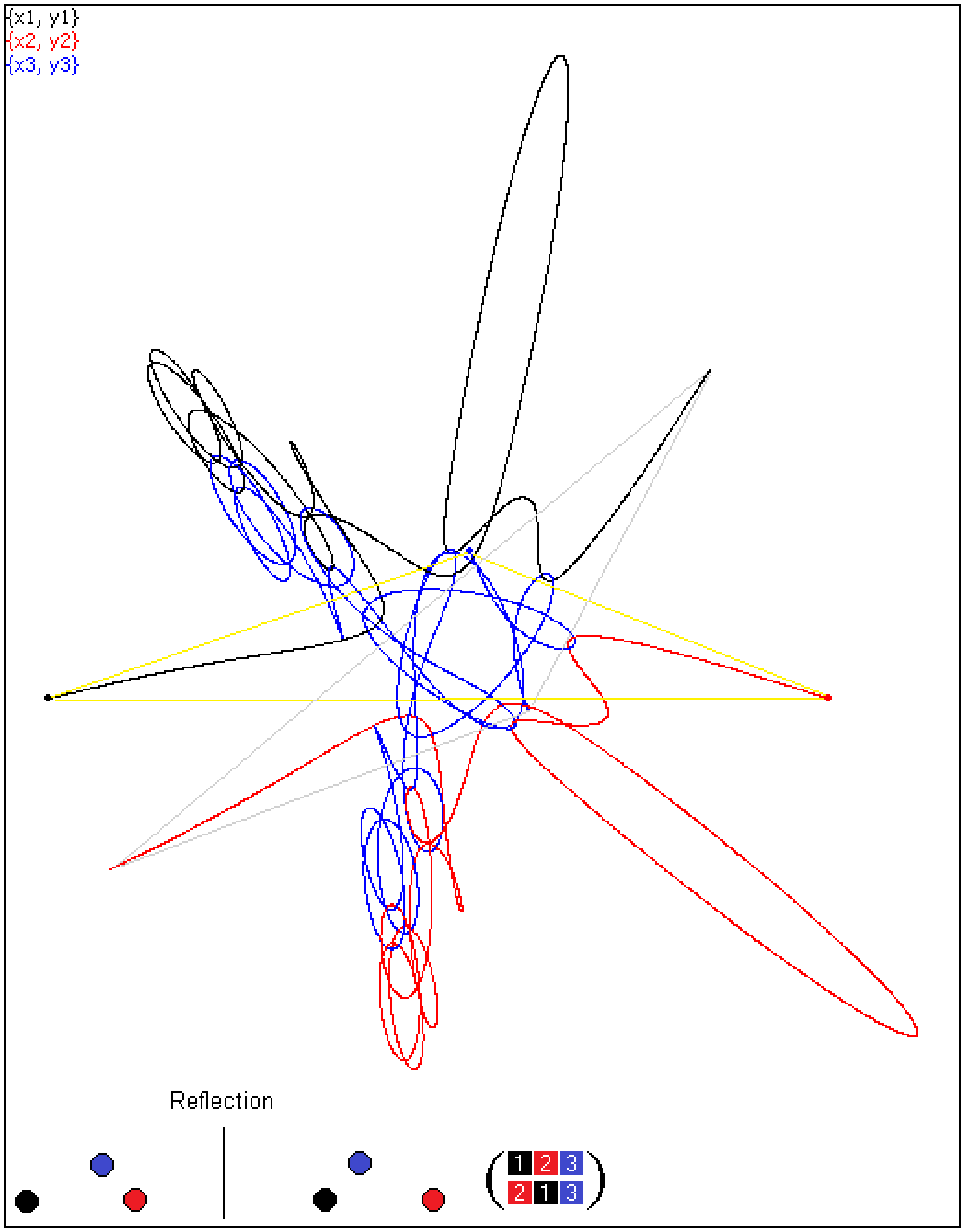
However, while watching those 30 simulations, a few of them gave an impression as though the triangle of the initial formation and the second one seemed similar. In order to verify if this visual effect is real, we modified the software for computing lengths of edges and the angles in the triangles. The numeric results [5] confirmed, that the visual impression of the similarity was real, being in fact *congruency* of the triangles in all the 12 special cases.

For example, in the simulation #18 the second triangle and the first are congruent, in a relation of reflection, and the trajectories connect the corresponding vertexes (with the identity permutation $Id$).

| | | |
|---|---|---|
| $AB = 1$ | $AC = 0.718590490501491$ | $BC = 0.309097112311281$ |
| $A'B' = 1.00000000170251$ | $A'C' = 0.718590493477556$ | $B'C' = 0.309097111598536$ |
| $\frac{A'B'}{AB} = 1.00000000170251$ | $\frac{A'C'}{AC} = 1.00000000414153$ | $\frac{B'C'}{BC} = 0.999999997694106$ |



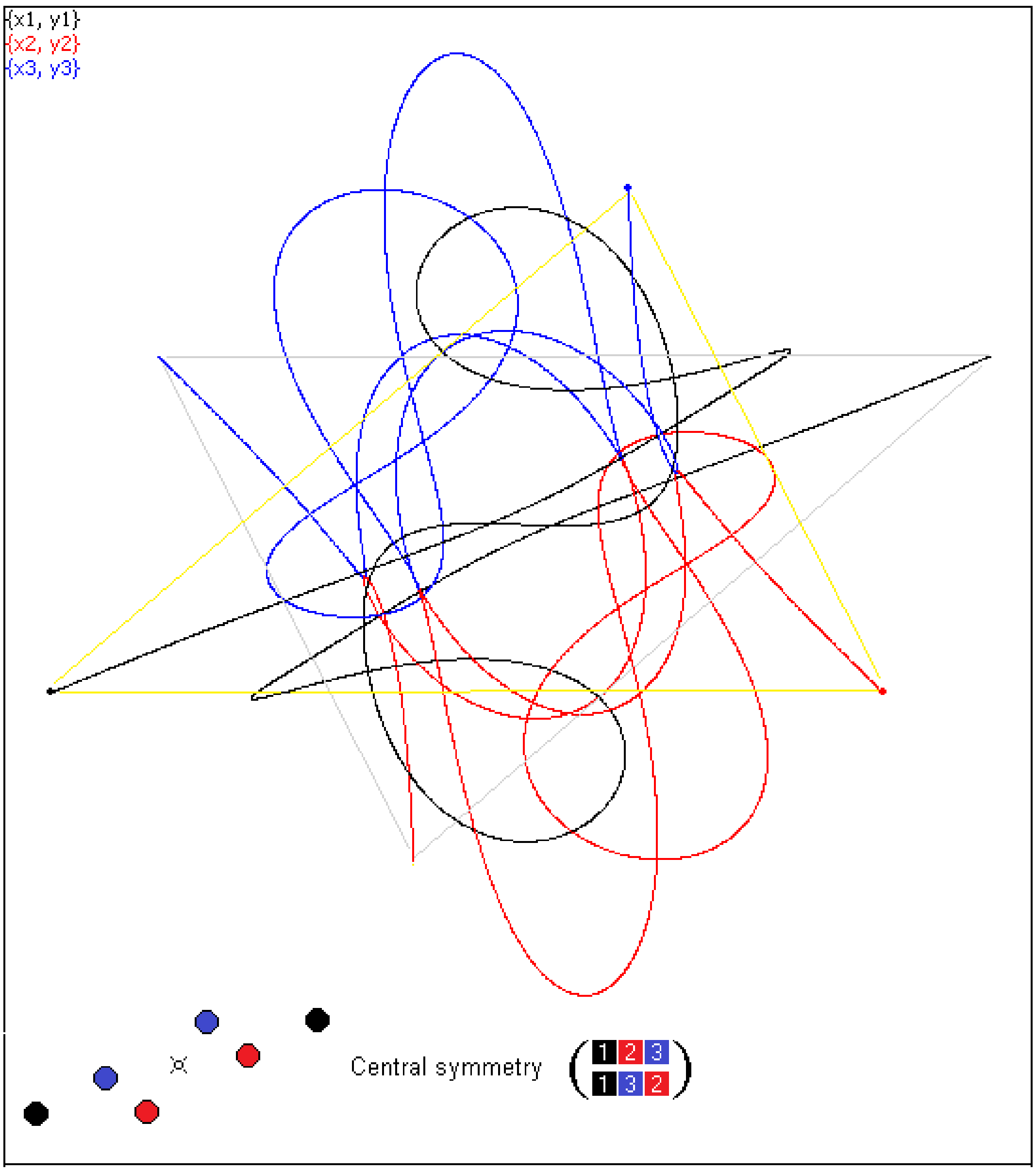

Simulation #18: the second and the first triangle are in a relation of reflection, no permutation.




Simulations 19: the triangles are in a relation of reflection with permutation $\binom{123}{213}$

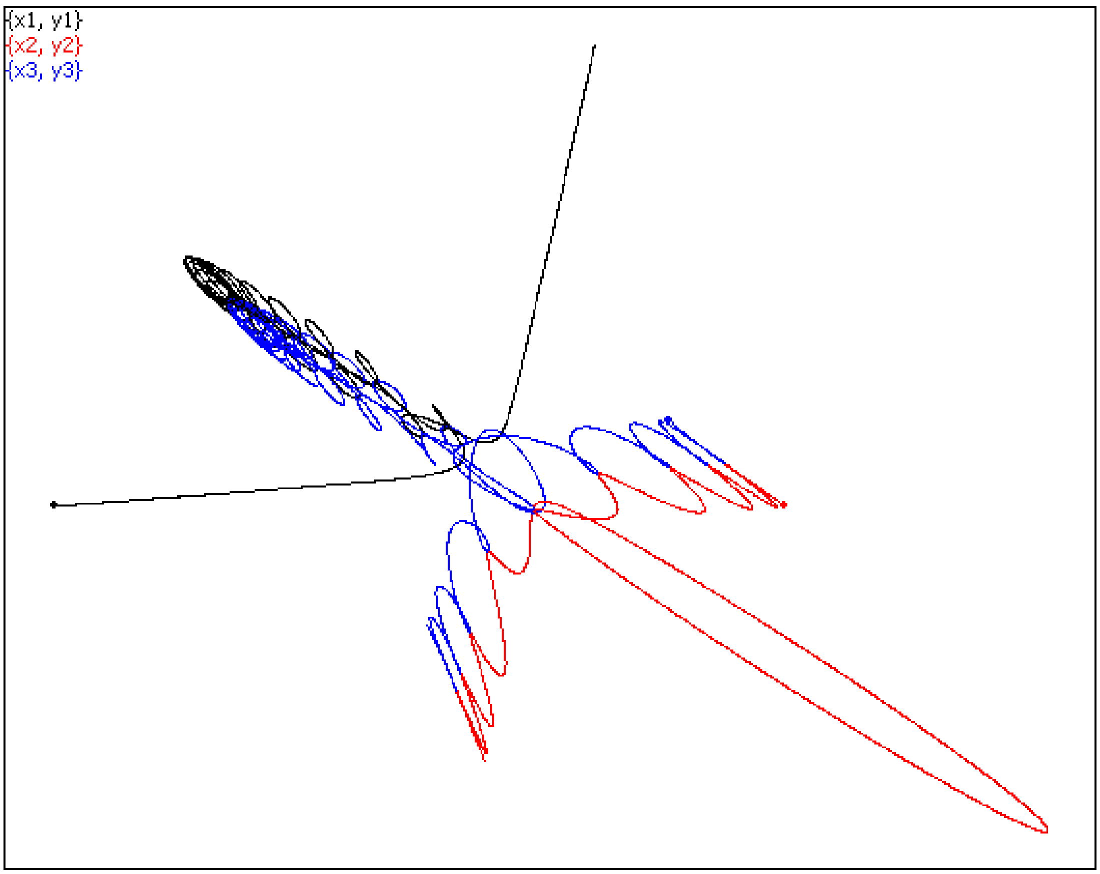
The properties of the simulation #14 are even more remarkable: the second triangle is congruent to the first as a result of 180° rotation. I.e. the triangles are in a relation of the central symmetry. At that, the trajectories connect not the corresponding vertexes, but effectuate a permutation $\binom{123}{132}$.

| | | |
|---|---|---|
| $AB = 1$ | $AC = 0.921456266427119$ | $BC = 0.679289996641939$ |
| $A'C' = 0.999999999906395$ | $A'B' = 0.921456266288986$ | $B'C' = 0.679289996759996$ |
| $\frac{A'C'}{AB} = 0.999999999906395$ | $\frac{A'B'}{AC} = 0.999999999850092$ | $\frac{B'C'}{BC} = 1.00000000017379$ |



Simulation #14: the triangles are in a relation of central symmetry with permutation $\binom{123}{132}$ .

Finally the simulation 30 which does not belong to the remarkable subset of the 12. It is brought to demonstrate a kind of "approximate symmetry" in some of the 30 orbits.




Simulation 30: "near" reflection, whose "symmetry" is "skewed", and no congruency takes place.

In the following section we are to analyze the numeric data [5] confirming the new properties.

# 6 The supporting data obtained via the Taylor Center software

As the new properties so far were confirmed only numerically, it's necessary to explain the numeric process employed for that purpose.

The integration method for the 30 simulations used here was the same modern Taylor method which was used by the authors [2] calling it Clean Numerical Simulation (CNS) at their super-computer. As they wrote, their implementation of the Taylor method admits arbitrary order and double precision. (They, however, didn't mention which order they used, and what the *double precision* means at their super computer.)

In this Taylor Center software (also admitting an arbitrary order) we used the order 30 with maximum precision of float point numbers supported by the processors Intel, which is the 10 byte float type called *extended*: 63-bit mantissa and 16-bit exponent.

For each of the 30 computed simulations the outputted actual data is provided under link [5] being comprised of the following elements:

- The header containing the sequence # of the simulation, its half-period (taken from [1]), and the number of integration steps;

- The initial lengths of the three edges of the triangle denoted ao1, ao2, ao3 at $t = 0$, when body #1 is at vertex $A$, body #2 - at vertex $B$, body #3 - at vertex $C$.

- ao1 (connecting bodies #2 and #3) opposite to vertex $A-$ the initial position of the body #1 (not necessarily moving towards $A'$);

- ao2 (connecting bodies #1 and #3) opposite to vertex $B-$ the initial position of the body #2 (not necessarily moving towards $B'$);

- ao3 (connecting bodies #1 and #2) opposite to vertex $C-$ the initial position of the body #3 (not necessarily moving towards $C'$).

- The lengths a1, a2, a3 of the three edges of the triangle at the second full stop at the moment $T/2$ :

- a1 (connecting bodies #2 and #3);

- a2 (connecting bodies #1 and #3);

- a3 (connecting bodies #1 and #2).

- The "best ratios" between lengths of the edges, chosen among $3! = 6$ possible permutations: "the best" meaning those triplets for which the three ratios are closest to a same value. If neither of the 6 permutations yields three ratios close to the same value, the message "no similarity" appears.

- The 6 values of components of the velocities in the moment $T/2$ of the second full stop posted as a proof of reaching the state of full stop and characterizing the achieved accuracy of the full stop.

If similarity takes place, the data package contains also:

- The proportions yielding approximately the same values, for example

a3/ao1 = 1.00000000202946
a2/ao2 = 0.999999997920113
a1/ao3 = 1.00000000030637.
Here all three proportions are approximately 1 demonstrating congruency.

**Remark 1** *Neither of the* 30 *cases demonstrated similarity with the proportion other than* 1*.*

- The $3 \times 3$ matrix of angles between the edges ao1, ao2, ao3 and edges a1, a2, a3 in order to see if there are angles close to 0° or 180°, for example

| | ao1 | ao2 | ao3 |
|---|---|---|---|
| a1 | 138.847783298464° | 179.999999953788° | 104.361663992048° |
| a2 | 179.999999940341° | 138.847783191836° | 63.2094472309557° |
| a3 | 116.790552658371° | 75.6383359097635° | $5.33608528907246e-008°$ |

- The respective permutation.

If the angles close to 0° or 180° are detected, the pairs of parallel edges are displayed. In the matrix above, such are ao2||a1, ao1||a2, ao3||a3.

# 7 Notes on accuracy

We see that the ratios expected to be 1, and the velocities expected to be 0, actually differ from the targeted values. The accuracy of those values depend on several factors: on the accuracy of the parameters provided by the author [1], and on the limits of accuracy in this Taylor integrator.

In the Taylor Center software the accuracy of integration (in ideal cases) may achieve up to 63 correct binary digits of the mantissa at every step, which corresponds to 18 correct decimal digits. Even with such ultimate 63-bit accuracy achievable at *one step*, the global error increases with growing number of steps (due to the rounding errors, or worse, due to catastrophic subtraction error in some problems). For example, in a test for simulation #1 integrated from 0 to its period $T$ and back to 0, the accuracy of the method in terms of the absolute error was: about $10^{-13}$ for the positions, and $10^{-12}$ for the velocities in 2500 integration steps.

Thinking about the reasons that the actual accuracy of the proportions and velocities obtained in this numerical experiment was not so good, first comes to mind that the values of the initial positions and the periods were specified by the authors [1] only up to 11 decimal digits (instead of possible 18 in the PCs).

# 8 Conclusions.

We have considered the 30 orbits found in [1] of periodic free fall of 3 equal masses. These orbits were obtained in an extensive search procedure, set with the only goal to satisfy the criterion of periodicity (2). However, it appeared that some of these orbits possess also other remarkable properties never targeted in the search criterion. Namely, some of the orbits were approximately symmetric (for example Simulation #30), while 12 orbits happened to be symmetric exactly (despite that the search goal was not set to find orbits with symmetry).

As Figure 1 of scattered initial positions for the point $\mathbf{q}_3$ demonstrates, there is no noticeable pattern in those 30 points.

Given a *bounded area* for searching $\mathbf{q}_3$, the following questions arise:

1. Is the number $N$ of initial points $\mathbf{q}_3$ specifying all periodic free fall orbits finite or infinite? (Number $N$ includes the already known 30 orbits).

2. Is the number $N_s$ of initial points $\mathbf{q}_3$ specifying all symmetrical periodic free fall orbits finite or infinite? If $N_s$ is finite, the set of such initial points $\mathbf{q}_3$ is of measure zero so that the probability of finding those 12 initial values $\mathbf{q}_3$ in a random search process is zero too. Then how could it happen that those 12 special orbits were found?

This enigmatic situation begs for an explanation and further research.

# References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2. Collisionless periodic orbits in the free-fall three-body problem. Xiaoming Li, Shijun Liao, (2018), https://arxiv.org/pdf/1805.07980.pdf

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6. Resources for the concept and discovery of Choreographies. http://taylorcenter.org/Simo/

7. Resources for the periodic solutions of a plane 3 body problem http://taylorcenter.org/Hudomal/

8. Resources for the free fall periodic solutions of plane 3 body problem. http://taylorcenter.org/XiaomingLI-ShijunLIAO/

9. The gallery of 12 high resolution orbits with special properties: http://taylorcenter.org/Workshops/3BodyFreeFall/Congruence/

## Appendix: the software and installation

A detailed outline of the Taylor Center software may be found here [4]. In it the hot link for downloading the software is:

http://taylorcenter.org/Gofen/TaylorCenterDemo.zip .

Download and unzip the file ("*Save*", don't "*Open*" it in your browser). Unzip and keep it in an empty folder of your choice, *TCenter.exe* being the only executable to run. Preserve this file and sub-folders structure (in order that the program work properly).

In the program you have to distinguish the *Main* (or *Front*) window, and the *Graph* window (which displays trajectories). Within the *Main* window there are 4 tabbed pages: *Equation setting*, *Debugging*, *Integration setting*, and *Graph setting*. When you load a script (from Demo or from a script file), you immediately get into the *Graph* window to play with the loaded simulation.

The initial values for 30 free fall periodic solutions were entered into the Demo of the Taylor Center, each of which can be loaded and played. Here is how.

1. Go to *Demo/3 bodies/Periodic (free fall)/Run simulation #...*

2. Enter a number of the desired sample between 1 and 30 in a small window (top left)[1].

3. When the integration reaches the termination point (the period $T$), the program displays the message. As you click OK, the program opens the *Graph* window displaying the entire trajectory. You may wish to *Play* it dynamically: by default the play duration is 25 s. Depending on the complexity of the curve, it may require something like 60-80 seconds. Enjoy the show, and then repeat everything from step 2 for another sample.

[1] The program loads the ODEs for the 3 body problem with the initial values corresponding to the selected periodic orbit, compiles, and integrates the problem until reaching the termination point – the period of this simulation entered from its file. (The period of the orbit is visible also in the Front panel in the *Constant* section as a comment line for constant $a$).