Exploratorium: a library of real time simulations for applied ODEs. The shape sphere: a remarkable transformation

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Abstract

This article refers to the special transformation introduced several decades ago [2, 3] used in studies of the 3 body problem. Not delving into the mathematical essence of it (explained in the sources [2, 3]), here we are to demonstrate the main properties and visualization of this transform utilizing an advanced ODE solver called the Taylor Center [1], capable to integrate initial value problems with high accuracy and to display solutions as real time animations in 2D and 3D stereo (viewable via red/blue glasses).

These unique graphical features are particularly instrumental for demonstrating the behavior of the shape sphere transform on the unit sphere.

Introduction.

This special transformation introduced several decades ago [2, 3] applies specifically for the plane 3 body Newtonian problem with point masses $m_1,\ m_2,\ m_3$

$$
\begin{aligned}
\ddot{x}_1 &= m_3(x_3 - x_1)r_{31} - m_2(x_1 - x_2)r_{12} \\
\ddot{y}_1 &= m_3(y_3 - y_1)r_{31} - m_2(y_1 - y_2)r_{12} \\
\ddot{x}_2 &= m_1(x_1 - x_2)r_{12} - m_3(x_2 - x_3)r_{23} \\
\ddot{y}_2 &= m_1(y_1 - y_2)r_{12} - m_3(y_2 - y_3)r_{23} \\
\ddot{x}_3 &= m_2(x_2 - x_3)r_{23} - m_1(x_3 - x_1)r_{31} \\
\ddot{y}_3 &= m_2(y_2 - y_3)r_{23} - m_1(y_3 - y_1)r_{31}
\end{aligned}
\tag{1}
$$

at the initial positions $(x_1,\ y_1) = \mathbf{q}_1,\ (x_2,\ y_2) = \mathbf{q}_2,\ (x_3,\ y_3) = \mathbf{q}_3$, where

$$
\begin{aligned}
r_{12} &= ((x_1 - x_2)^2 + (y_1 - y_2)^2)^{-3/2} \\
r_{23} &= ((x_2 - x_3)^2 + (y_2 - y_3)^2)^{-3/2} \\
r_{31} &= ((x_3 - x_1)^2 + (y_3 - y_1)^2)^{-3/2}.
\end{aligned}
$$

The shape sphere transform $(\mathbf{q}_1,\ \mathbf{q}_2,\ \mathbf{q}_3) \longmapsto (u_1,\ u_2,\ u_3)$ maps triplets of initial points $(\mathbf{q}_1,\ \mathbf{q}_2,\ \mathbf{q}_3)$ into the points $(u_1,\ u_2,\ u_3)$ of the unit sphere $u_1^2+u_2^2+u_3^2=1$ (called the shape sphere) by the following formulas. First we obtain the so called Jacobi coordinates $\boldsymbol{\rho}_1,\ \boldsymbol{\rho}_2$

$$\begin{aligned} \boldsymbol{\rho}_1 &= (\mathbf{q}_1-\mathbf{q}_2)\sqrt{\frac{m_1 m_2}{m_1+m_2}} \\ \boldsymbol{\rho}_2 &= \sqrt{\frac{m_3(m_1+m_2)}{m_1+m_2+m_3}}\left(\mathbf{q}_3-\frac{m_1\mathbf{q}_1+m_2\mathbf{q}_2}{m_1+m_2}\right), \end{aligned} \tag{2}$$

$\mathbf{q}_1,\ \mathbf{q}_2,\ \mathbf{q}_3,\ \boldsymbol{\rho}_1,\ \boldsymbol{\rho}_2 \in \mathbf{E}^2$. Then - the values

$$\begin{aligned} w_1 &= \frac{|\boldsymbol{\rho}_1|^2-|\boldsymbol{\rho}_2|^2}{2} \\ w_2 &= (\boldsymbol{\rho}_1,\ \boldsymbol{\rho}_2) \\ w_3 &= |\boldsymbol{\rho}_1\times\boldsymbol{\rho}_2| \end{aligned} \tag{3}$$

called the shape *space* $(w_1,\ w_2,\ w_3)\in\mathbf{E}^3$. Finally, the points $(u_1,\ u_2,\ u_3)$ on the unit sphere

$$u_i = \frac{w_i}{|\mathbf{w}|}, \quad i=1,\ 2,\ 3 \tag{4}$$

called the shape *sphere.*

Notation 1 *There are two distinct concepts: the* shape sphere *and the* shape space*. We denote the points of a unite sphere called the shape sphere* $(u_1,\ u_2,\ u_3)$*, and the points of shape space* $(w_1,\ w_2,\ w_3)$*. Here is the table of correspondence of notations in different authors:*

	This paper	*Kerrigan [2]*	*Reichert [3]*	*Montgomery [4]*
Shape space	$w_1,\ w_2,\ w_3$	$w_1,\ w_2,\ w_3$	$w_1,\ w_2,\ w_3$	$w_1,\ w_2,\ w_3$
Shape sphere	$u_1,\ u_2,\ u_3$	$w_1',\ w_2',\ w_3'$		

Correspondingly, we will use expressions w*-transform and* w*-points vs.* u*-transform* u*-points.*

By its construction, the points of the shape sphere have the fundamental property expressed in the following Theorem.

Theorem 1 *All triplets of points* $(\mathbf{q}_1,\ \mathbf{q}_2,\ \mathbf{q}_3)$*,* $\mathbf{q}_i\in\mathbf{E}^2$ *having respectively the same masses* $m_1,\ m_2,\ m_3$*, which comprise similar triangles*[1] *in the* same orientation *map into one* u*-point* $(u_1,\ u_2,\ u_3)$ *on the shape sphere. The points making the similar triangles in the opposite orientation map into the point* $(u_1,\ u_2,\ -u_3)$ *on the shape sphere. The triplets representing syzygies map onto the equator* $u_3=0$ *of the sphere.*

[1] I.e. triangles $\triangle ABC$ with the *same proportions* between the edges $AB,\ AC,\ BC$ and with the *same angles* $\alpha,\ \beta,\ \gamma$.

Proof 1 *Let complex numbers* $z_1,\ z_2,\ z_3$ *represent a triangle* $\triangle ABC$*, while* $\bar z_1,\ \bar z_2,\ \bar z_3$ *represent a reflection of* $\triangle ABC$ *about the abscissa i.e. the triangle in the opposite orientation. Then* $c+kz_1,\ c+kz_2,\ c+kz_3$ *, where* c *and* k *are complex numbers, represent a triangle* $\triangle A'B'C'$ *similar to* $\triangle ABC$ *(possibly translated and turned). At that, if* $k = re^{i\alpha}$*, then* α *means a turn of* $\triangle ABC$ *by the angle* α*, and* r *the stretch of it.*

1. *Similarity. Apply formula (2) for the* $\triangle A'B'C'$ *obtaining* $\boldsymbol{\rho}'_1,\ \boldsymbol{\rho}'_2$ *for the* $\triangle A'B'C'$

$$\begin{aligned}\boldsymbol{\rho}'_1 &= (c+kz_1-c-kz_2)\sqrt{\frac{m_1m_2}{m_1+m_2}}\\ \boldsymbol{\rho}'_2 &= \left(c+kz_3-\frac{m_1(c+kz_1)+m_2(c+kz_2)}{m_1+m_2}\right)\sqrt{\frac{m_3(m_1+m_2)}{m_1+m_2+m_3}},\end{aligned}$$

so that

$$\begin{aligned}\boldsymbol{\rho}'_1 &= k\boldsymbol{\rho}_1\\ \boldsymbol{\rho}'_2 &= k\boldsymbol{\rho}_2.\end{aligned}$$

Similarly obtain $w'_1,\ w'_2,\ w'_3$ *for the* $\triangle A'B'C'$ *keeping in mind that* $|k| = r$ *:*

$$w'_i = w_i r^2, \quad i = 1,\ 2,\ 3. \tag{5}$$

This demonstrates, that the triplet $w_1,\ w_2,\ w_3$ *does not depend on turn* α *any more though still depending on the stretch* r*. However, these values* $w'_1,\ w'_2,\ w'_3$ *being normalized to the unit sphere by formula (3), lose the factor* r^2 *so that* $u'_i = u_i, \quad i = 1,\ 2,\ 3,$ *meaning that the similar triangles in the same orientation map into one point* $(u_1,\ u_2,\ u_3)$*.*

2. *Reflection. Apply formula (2) for* $\bar z_1,\ \bar z_2,\ \bar z_3$ *noticing that*

$$\begin{aligned}\boldsymbol{\rho}_1(\bar z_1,\ \bar z_2,\ \bar z_3) &= \overline{\boldsymbol{\rho}_1(z_1,\ z_2,\ z_3)} = \bar{\boldsymbol{\rho}}_1\\ \boldsymbol{\rho}_2(\bar z_1,\ \bar z_2,\ \bar z_3) &= \overline{\boldsymbol{\rho}_2(z_1,\ z_2,\ z_3)} = \bar{\boldsymbol{\rho}}_2.\end{aligned}$$

Spell out the scalar and vector products. If

$$\begin{aligned}\boldsymbol{\rho}_1 &= a_1+ib_1\\ \boldsymbol{\rho}_2 &= a_2+ib_2\end{aligned}$$

then

$$\begin{aligned}(\boldsymbol{\rho}_1,\ \boldsymbol{\rho}_2) &= a_1a_2+b_1b_2\\ |\boldsymbol{\rho}_1\times\boldsymbol{\rho}_2| &= \begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}.\end{aligned}$$

Therefore, according to (3), the conjugation (changing the sign of $b_1,\ b_2$*) does not affect* $u_1,\ u_2$ *, but changes the sign of* u_3 *to the opposite.*

3. *Syzygy, i.e.* $\mathbf{q}_1,\ \mathbf{q}_2,\ \mathbf{q}_3$ *are collinear. Consider a line* l *made by* $\mathbf{q}_1,\ \mathbf{q}_2$*. Observe that* $\boldsymbol{\rho}_1$ *is proportional to the vector* $\mathbf{q}_1 - \ \mathbf{q}_2$*. Analyzing* $\boldsymbol{\rho}_2$*, observe that* $\dfrac{m_1\mathbf{q}_1 + m_2\mathbf{q}_2}{m_1 + m_2}$ *is a point on* l*,* $\mathbf{q}_3$ *belongs to* l *too, therefore* $\boldsymbol{\rho}_2 || \boldsymbol{\rho}_1$ *so that* $\omega_3 = 0$ *in (3).*

Let's reformulate this Theorem for points $w_1,\ w_2,\ w_3$ of the shape space in which the similarity must be replaced with congruency of triangles.

Theorem 2 *All triplets of points* $(\mathbf{q}_1,\ \mathbf{q}_2,\ \mathbf{q}_3),\quad \mathbf{q}_i \in \mathbf{E}^2$ *having respectively the same masses* $m_1,\ m_2,\ m_3$*, which make* congruent *triangles in the* same orientation *map into one w-point* $(w_1,\ w_2,\ w_3)$ *in the shape space. The points making the congruent triangles in the opposite orientation map into the point* $(w_1,\ w_2,\ -w_3)$ *in the shape space. The triplets representing syzygies map onto the plane* $w_3 = 0$*.*

Corollary 1 *If the three bodies move keeping the formation of similar triangles (though possibly changing the sizes), they map into one u-point, but into different w-points belonging to a particular ray with the origin in the center of the shape sphere (follows from (5)).*

Exercise 1 *Consider an equilateral triangle, say* $z_1 = -1,\quad z_2 = 1,\quad z_3 = i\sqrt{3}$ *for the equal masses* $m_1 = \ m_2 = \ m_3$*. Applying formulas (2), check that for this equilateral triangle* $|\boldsymbol{\rho}_1|^2 = |\boldsymbol{\rho}_2|^2$ *and* $\boldsymbol{\rho}_1 \perp \boldsymbol{\rho}_2$ *so that (according to (3)),* $w_1 = w_2 = 0$*, and consequently* $u_1 = u_2 = 0$*, while* $u_3 = \pm 1$*. This means that all equilateral triangles map onto the North or South poles of the shape sphere.*

After this piece of theory, let's get back to our goal: to visualize the behavior of this transform utilizing the dynamical graphics (real time animation) of the Taylor Center software [1]. Unlike a static graph, this real time animation allows to display the dynamic of a specific motion: in conventional plane (in 2D) and on the shape sphere in 3D stereo.

The computer representation.

In all the simulations below the ODEs (1) in computer representation take form:

t' = 1
x1' = vx1
y1' = vy1
x2' = vx2
y2' = vy2

```
x3' = vx3
y3' = vy3
vx1' = m3*dx31*r31 - m2*dx12*r12
vy1' = m3*dy31*r31 - m2*dy12*r12
vx2' = m1*dx12*r12 - m3*dx23*r23
vy2' = m1*dy12*r12 - m3*dy23*r23
vx3' = m2*dx23*r23 - m1*dx31*r31
vy3' = m2*dy23*r23 - m1*dy31*r31
```

preceded by the auxiliary variables and equations like

```
r31 = (dx31^2 + dy31^2)^m15
```

(where m15 = -1.5).

In order to visualize the shape sphere motion, we have to add a few constants into the section of constants

```
sqm1m2 = sqrt(m1*m2/(m1 + m2) )
sqm1m2m3 = sqrt(m3*(m1 + m2)/(m1 + m2 + m3 ) )
m1plm2 = m1 + m2
```

and the following equations into the auxiliary section:

```
ro1x = sqm1m2*(x1 - x2)
ro1y = sqm1m2*(y1 - y2)
ro2x = sqm1m2m3*(x3 - (m1*x1 + m2*x2)/m1plm2)
ro2y = sqm1m2m3*(y3 - (m1*y1 + m2*y2)/m1plm2)
w1 = (ro1x^2 + ro1y^2 - ro2x^2 - ro2y^2)*0.5
w2 = ro1x*ro2x + ro1y*ro2y
w3 = ro1y*ro2x - ro1x*ro2y
r = sqrt(w1^2 + w2^2 + w3^2)
u1 = w1/r
u2 = w2/r
u3 = w3/r
```

encoding the formulas of the shape sphere transform. All of them are included into the respective scripts analyzed in the next section.

Simulations visualizing the properties of the shape sphere.

Let's begin with the classical Lagrange and Euler cases. In the Lagrange case the three bodies move in a formation of equilateral triangles which rotate and (generally) change the size, yet all mapping into one point of the shape sphere (by the Theorem 1) which is the North or South Poles (according to the Exercise).

In the Euler case the three bodies move in syzygy mapping into a point of the equator of the shape sphere.

Those changing formations (triangles or lines), however, map into one u-point of the shape sphere as long, as the process of numeric integration preserves the high enough accuracy of the solutions of the Lagrange and Euler cases (i.e. as long as the solution preserves the particular formation). However, after several cycles, the accuracy lowers, and the integration errors disturb the initial formation making it chaotic. When this happens, we see how the so far steady u-point of the shape sphere (corresponding to the initial formations) jumps into a motion on the surface of the sphere as soon as the three body slide into the chaotic motion.

In the simulations below typically it is recommended first to see the dynamic of conventional motion in a 2D plane, then - the corresponding motion on the unit sphere in 3D, or both. (All 3D simulations must be viewed either using the Red/Blue anaglyph glasses for full stereo perception, or in the axonometric mode without glasses - see the Appendix).

The Lagrange case for n equal masses.

The equal masses are placed at the vertices of the planar regular polygon. The vectors of their velocities must belong to a regular polygon in the same plane too being of the same absolute value. This value determines which type of the conic sections (ellipse, parabola, or hyperbola) all of the bodies follow. Moving along these conic sections, the bodies stay in the formation similar to the initial polygon, which may turn and stretch. In order to play with n body Lagrange case, go to *Set n-body problem*, *Planar*, *Elliptic* in the Taylor Center program.

Back to our 3-body study, load a 2D script file *LagrangeEqMasses.scr* . Click Play and watch, how the three bodies first perform the elliptic Kepler motion in the formation of similar equilateral triangles. Then, after several loops, the formation is getting disturbed into chaotic motion.

Now load a 3D file *LagrangeEqMassesBoth.scr* displaying the same motion of the 3 bodies plus the point $(u_1,\ u_2,\ u_3)$ on the North pole of the shape sphere. Watch how in accordance with the theory, the point $(u_1,\ u_2,\ u_3)$ stays at the North pole steady until the very end, when this point jumps as soon as the bodies slide into chaotic motion.

Finally load and play a 3D file *LagrangeEqMassesw1w2w3.scr* . Now, along with the planar motion of the 3 bodies, you can see the an oscillation of the point $(w_1,\ w_2,\ w_3)$ along a straight line (a vertical ray) synchronized with the stretching of the equilateral triangle: as long as the motion of the bodies remains regular.

The Lagrange case for 3 unequal masses $m_1,\ m_2,\ m_3$.

The Lagrange case for unequal masses is possible only for $n = 3$ and is less intuitive. The three bodies also move along the conic sections of the same type preserving the formation of equilateral triangles. The initial values are computed by the following formulas.

The initial equilateral triangle

$$\begin{aligned} a_1 &= 0.5, \quad b_1 = 0 \\ a_2 &= 0, \quad\ \ b_2 = \frac{\sqrt{3}}{2} \\ a_3 &= -0.5, \quad b_3 = 0. \end{aligned}$$

Then it's center of masses

$$\begin{aligned} c_x &= \frac{m_1 a_1 + m_2 a_2 + m_3 a_3}{m_1 + m_2 + m_3} \\ c_y &= \frac{m_1 b_1 + m_2 b_2 + m_3 b_3}{m_1 + m_2 + m_3}. \end{aligned}$$

Then the initial positions

$$\begin{aligned} x_1 &= a_1 - c_x \\ y_1 &= b_1 - c_y \\ x_2 &= a_2 - c_x \\ y_2 &= b_2 - c_y \\ x_3 &= a_3 - c_x \\ y_3 &= b_3 - c_y. \end{aligned}$$

Then

$$\begin{aligned} r_1 &= \sqrt{x_1^2 + y_1^2} \\ r_2 &= \sqrt{x_2^2 + y_2^2} \\ \alpha_1 &= \arcsin \frac{y_1}{r_1} \\ \alpha_2 &= \arcsin \frac{y_2}{r_2}. \end{aligned}$$

And finally, with an arbitrary chosen parametric velocity v, the initial velocities are

$$
\begin{aligned}
v_{x1} &= v r_1 \cos(\alpha_1 + \pi/2) \\
v_{y1} &= v r_1 \sin(\alpha_1 + \pi/2) \\
v_{x2} &= v r_2 \cos(\alpha_2 + \pi/2) \\
v_{y2} &= v r_2 \sin(\alpha_2 + \pi/2) \\
v_{x3} &= -\frac{m_1 v_{x1} + m_2 v_{x2}}{m_3} \\
v_{y3} &= -\frac{m_1 v_{y1} + m_2 v_{y2}}{m_3}
\end{aligned}
$$

where value v determines eccentricities of the conic sections.

In order to obtain the circular Lagrange motion, v must be calculated in the following way:

$$
\begin{aligned}
r_{12} &= \left((x_1 - x_2)^2 + (y_1 - y_2)^2\right)^{-3/2} \\
r_{31} &= \left((x_3 - x_1)^2 + (y_3 - y_1)^2\right)^{-3/2} \\
v &= \sqrt{\frac{(m_3(x_3 - x_1)r_{31} - m_2(x_1 - x_2)r_{12}))}{r_1 \cos\alpha_1}}
\end{aligned}
$$

which is the condition that the centrifugal force acting at a body is compensated with the gravity force.

Load a 2D script file *LagrangeUneqMasses.scr* . Click Play and watch, how the three bodies first perform the Kepler motion along ellipses of different sizes, yet preserving the formation of similar equilateral triangles. Then, after several loops, the formation disturbs into chaotic motion.

Now load a 3D file *LagrangeUneqMassesBoth.scr* displaying the same motion of the 3 bodies plus the point $(u_1,\ u_2,\ u_3)$ on the shape sphere. Watch how in accordance with the theory, the point $(u_1,\ u_2,\ u_3)$ stays steady until the very end, when it jumps as soon as the bodies slide into chaotic motion. Observe that now, unlike in the case of equal masses above, $(u_1,\ u_2,\ u_3)$ stays not at the North pole - because even though the triangles are still equilateral, the masses are not equal.

Load a 3D file *LagrangeUneqMassesuw1uw2uw3.scr* to watch simultaneously $(u_1,\ u_2,\ u_3)$ and $(w_1,\ w_2,\ w_3)$.

Then load a 3D file *LagrangeUneqMassesxyzuv.scr* to watch simultaneously the bodies, $(u_1,\ u_2,\ u_3)$ and $(w_1,\ w_2,\ w_3)$. Observe, that now the w-points oscillate along a ray declined to the horizon.

In the simulations above the bodies move in the formation of equilateral triangles whose size changes. There exists, however, a special setting when the conic section type of the trajectories is circular so that the triangles remain congruent (during the regular phase of the motion). Therefore as long as the triangles remain congruent, they map into one w-point (rather than into a ray).

Load a 2D file *LagrangeCircUneqMasses.scr* to watch the circular motion in a plane.

Then load a 3D file *LagrangeCircUneqMassesw1w2w3.scr* displaying how the w-point is steady (rather than oscillating) until it jumps into chaotic motion.

Finally, consider a simulation of a disturbed Lagrange case where the state of chaos happens almost immediately: load and play a 2D file *3Bod9995.scr* . At the beginning the bodies make a piece of ellipse falling into chaotic dance after a short while.

Now load and play a 3D file *3Bod9995u1u2u3.scr* displaying the respective motion of $(u_1,\ u_2,\ u_3)$. Watch that this point too remains steady for short while, and then jumps into a random motion on the surface of the sphere.

The Euler case for masses $m_1,\ m_2,\ m_3$

The Euler case is such that the three body move preserving the collinear formation. The initial setting for this case requires that the proportions between the masses $m_1,\ m_2,\ m_3$ and the distances between them satisfy special equations.

The initial values are computed by the following formulas. It is presumed that $y_1 = y_2 = y_3 = 0$, while

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = x_2 + r$$

where r may be defined arbitrarily ($r = 2$ in the following simulations). The value r must satisfy the polynomial equation of degree 5:

$$(m_1+m_2)r^5+(3m_1+2m_2)r^4+(3m_1+m_2)r^3-(m_2+3m_3)r^2-(2m_2+3m_3)r-(m_2+m_3) = 0.$$

However, with r chosen, we do not need to solve it. It suffices to arbitrarily choose m_1 and m_2 (we set $m_1 = 19$, $m_2 = 38$) and then

$$m_3 = \frac{m_1(r^5 + 3r^4 + 3r^3) + m_2(r^5 + 2r^4 + r^3 - r^2 - 2r - 1}{3r^2 + 3r + 1}.$$

Then the center of masses

$$c = \frac{x_1 m_1 + x_2 m_2 + x_3 m_3}{m_1 + m_2 + m_3}$$

and the velocities are defined as follows. Setting $v_{x1} = v_{x2} = v_{x3} = 0$ and an choosing an arbitrary (parametric) v_{y1} which affects the eccentricity of the ellipses or other conic sections (here $v_{y1} = 8$),

$$\begin{aligned} v_{y2} &= \frac{x_2 - c}{x_1 - c} v_{y1} \\ v_{y3} &= \frac{x_3 - c}{x_1 - c} v_{y1}. \end{aligned}$$

In order to obtain the circular Euler motion, v_{y1} must be calculated in the following way:

$$\begin{aligned} r_{12} &= \left((x_1-x_2)^2+(y_1-y_2)^2\right)^{-3/2} \\ r_{31} &= \left((x_3-x_1)^2+(y_3-y_1)^2\right)^{-3/2} \\ v_{y1} &= \sqrt{|(m_3(x_3-x_1)r_{31}-m_2(x_1-x_2)r_{12})(x_1-c))|} \end{aligned}$$

which is the condition that the centrifugal force acting at a body is compensated with the gravity force.

This completes the setting of the initial values for the Euler case.

Load a 2D script file *Euler.scr* , click Play and watch how at the beginning the bodies in a linear formation run along ellipses, until the formation is ruined at the end into chaotic motion.

Now load a 3D file *EulerBoth.scr* and click Play. At the beginning and almost to the end you can see the three bodies fulfilling the motion in a linear formation along ellipses plus the point $(u_1,\ u_2,\ u_3)$ on the equator of the shape sphere, where it remains steady as it should as long as the three bodies move regularly. However, when the bodies slide into chaotic motion, so does also the point $(u_1,\ u_2,\ u_3)$.

Finally load a 3D file *Eulerw1w2w3.scr* displaying the motion of the w-point for the Euler degenerated and congruent (rather than similar) triangles. Watch the oscillation of the w-point along a ray in the equatorial plane (till the formation ruins).

That was the elliptic motion.

Now load and Play first the 2D file *EulerCircular.scr* displaying the circular motion in a plane (until it breaks into a chaos).

Then load and Play a 3D file *EulerCircularw1w2w3.scr* displaying the motion of the w-point for the Euler degenerated and congruent (rather than similar) triangles. Now this w-point remains still (rather than oscillating) until the formation breaks into a chaos.

Three bodies free fall periodic motion [5].

Here we are to demonstrate one of newly discovered cases of free fall of three bodies having remarkable properties. In a typical random setting trajectories of three body motion are chaotic. That is why the discovery of the cases of periodic fall by Xiaoming Li and Shijun Liao [6] (2018) is so amazing. They have discovered hundreds of settings for resting three bodies of particular masses, which led to periodic motion: i.e. the bodies started at the given points of rest and returned back to their initial points after motion along sophisticated orbits during a period T. It's worth particular mentioning that ...

- In the discovered periodic trajectories the initial moment is not the only resting point. The number of moments of full stop in every such trajectory is exactly 2: namely the initial moment and the moment $T/2$ where all three bodies come to rest also at other points of space and generally in a different formation.

- The three bodies oscillate between a pair of the initial and second set of the rest points (a particular pair of sets for each simulation).

In the Taylor Center software you can watch 30 cases of such free fall simulations for the equal masses under *Demo/Three bodies/Periodic free fall*.

Among those 30 cases there are particularly remarkable ones when the two formations of the rest are *congruent* triangles in opposite orientation (while typically those two rest formations are unrelated triangles).

First load and play 2D file *XiaomingLIAO14.scr* to watch the free fall in its dynamic in a plane. This simulation displays how the points reach the second configuration of rest during a half-period of their motion.

Then load a 2D file *XiaomingLIAO14Return.scr* displaying how the bodies return into the initial position during the full period of the motion.

Now you can watch the respective motion of point $(u_1,\ u_2,\ u_3)$ on the shape sphere and also of the points $(w_1,\ w_2,\ w_3)$ in the shape space loading 3D files

- *XiaomingLIAO14Both.scr* - $(u_1,\ u_2,\ u_3)$ on the shape sphere;
- *XiaomingLIAO14w1w2w3.scr* - $(w_1,\ w_2,\ w_3)$ in the shape space.

Observe that in both cases u_3 or w_3 change the sign at the second rest point, as they should because the triangles are in the opposite orientation so that the points $(u_1,\ u_2,\ u_3)$ corresponding to the rest are in opposition, just as $(w_1,\ w_2,\ w_3)$ are.

The 8-shape choreography

Our final simulation is that of the 8-shape choreography: the simplest in the list of 345 (compiled by Carles Simo [7]). Choreography is such a formation when three bodies move along the same periodic curve one after the other. In the Taylor Center software you can watch all 345 of them under *Demo/Three bodies/Choreography*.

Load and play 2D file *Simo1.scr* displaying 8-shape choreography in plane.

Now load and play 3D file *Simo1Both.scr* displaying 8-shape choreography in plane and on the shape sphere.

Conclusion

We have reviewed the special transformations into the *shape sphere* and *shape space* introduced by the scholars studying the three body problem. The Taylor Center software happened to be particularly instrumental for visualizing the dynamic of motion both conventionally in a plane and on the shape sphere in 3D stereo.

References

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4. Richard Montgomery, "The Three-Body Problem and the Shape Sphere", 2015, http://dx.doi.org/10.4169/amer.math.monthly.122.04.299
5. Alexander Gofen, Three body free fall periodic orbits: new remarkable features, 2021, http://taylorcenter.org/Workshops/3BodyFreeFall/Index.htm
6. Xiaoming Li and Shijun Liao, Collisionless periodic orbits in the free-fall three-body problem. https://arxiv.org/pdf/1805.07980.pdf
7. Carles Simó, http://taylorcenter.org/Simo/

The software basics and installation

A detailed outline of the Taylor Center software may be found here [1]. The hot link in [1] for downloading it is:

http://taylorcenter.org/Gofen/TaylorCenterDemo.zip .

Download and unzip the file ("*Save*", don't "*Open*" it in your browser). Unzip and keep it in an empty folder of your choice, *TCenter.exe* being the only executable to run. Preserve this file and sub-folders structure (in order that the program work properly).

Then download a zipped structure of folders with script files of the simulations from here http://taylorcenter.org/Exploratorium/ShapeSphere.zip and unzip it into an empty folder of your choice, say *ShapeSphere*. This will be your folder to navigate from the program *TCenter.exe* in order to pick the necessary script and the file *SimulationsCatalog.rtf* containing the list of all script files in this Exploratorium.

In the program you have to distinguish the *Main* (or *Front*) window, and the *Graph* window (which displays trajectories). Within the *Main* window there are 4 tabbed pages: *Equation setting*, *Debugging*, *Integration setting*, and *Graph setting*. When you load a script, you immediately get into the *Graph* window to play with the loaded simulation. However, in order to explore the ODEs and parameters, you will have to visit the tabbed pages of the *Main* window.

Methods of loading and playing scripts

Here are several methods of loading scripts (occurring inside the pdf file of this Exploratorium) with various levels of automation.

1. No automation (when the Exploratorium is on paper). This is a general case (unrelated to this Exploratorium) when you wish to load scripts from any folder, say, *Samples* coming with the Taylor Center software. With this method, in order to load a particular script file (say, mentioned in the *print)*, go to *File/Load script* menu either in the *Main* or in *Graph* window, and navigate to the desired folder in the *open file* dialog box (the mode of displaying in the dialog box must be detailed list in alphabetical order). Examining this list, find the file mentioned in the Exploratorium. The moment you click *Open*, this file will be run displaying the final picture of the motion. Then, by clicking *Play* button, you can run the simulation and watch the motion in real time. Most simulation are in 3D stereo (on a black background) requiring a pair of red/blue glasses. Clicking the check box *Axonometry*, you can view them conventionally without the red/blue glasses just as other 2D images.

2. For this and the following methods, first load the catalog of all simulations of this textbook under *Demo/Load script list* menu item in the main window. There, in the *Open file dialog box* navigate to your folder *ShapeSphere* and in it open the file *SimulationsCatalog.rtf* so that you will see the list of all available simulations in the *Help window*. You can run each of them by double clicking the desired file at any place of the line (instead of navigating inside all required sub-directories in Method 1).

   More automation is available when this Exploratorium is a pdf file which you read at the same PC. While reading the pdf text from your screen, when you wish to run a particular simulation, first **select and copy the desired file name into the clipboard by your mouse**. This is a presumption for the Methods 3 and 4 below when the check box *Auto mode* is either unchecked, or checked.

3. In the Help window the check box *Auto mode* is unchecked. Having the script file of interest already copied into the clipboard, just click over the Help window activating it - and this action will trigger running of the selected script in the clipboard.

4. In the Help window the check box *Auto mode* is checked, meaning that the Taylor Center is in auto mode keeping to check whether the clipboard has changed every half second. In this fully automatic mode, in order to run every script of your interest mentioned in the pdf file, all that you need is to select it in the pdf reader and copy it into the clipboard. In a matter of a second you will get this file loaded and ready for playing in the Taylor Center.

If you plan to compare graphs of some two related simulations, you may load two instances of the *TCenter.exe* (especially if having a wide screen or two monitors). Having the two instances of the program loaded, you may watch still images at both, however do not initiate *Playing* simultaneously, because the real time playing function requires the entire resources of the PC exclusively.

After running a script from the Exploratorium, you may wish to make a change in the initial values and constants in order to see their effect. In order to do it, look into the editor panes for *Constants* and Initial values in the *Main* window, making the desired change. When a change is made, the Graph window disappears, and you need to *Compile* the modified problem. If the compilation succeeded (i.e. you did not introduce mistakes), you will see the *Graph* page. In it click the *Previous* button - which will bring you again to the Graph window ready to play the modified problem. After your changes, the *previous* setting of the sizes may happen to be not the best. Click the button *Adjust* which adjusts the sizes to create enough room for the image.

The optimal conditions for stereo viewing

In order to perceive the real stereo, you need a pair of reg/blue glasses (the left - red, the right - blue) putting them over your optical glasses used for reading screens. If you do not have red/blue glasses, you still can watch the simulations switching them into the mode of the conventional axonometric projection (the check-box Axonom).

In order to achieve the best stereo effect, the following setting should be made for your monitor and environment.

In a case of a desktop monitor here are the optimal settings...

1. Better have a monitor with black matte surface.
2. The light in the room must be as little as possible so that the surface of an inactive monitor look black.
3. Have a pair of Red/Blue glasses whose Red and Blue filters let through only the narrow spectrum of the respective colors. In order to test that your glasses satisfy this criterion, overlap two pairs of such glasses with the opposite filters over each other and look at the bright light. Ideally, the overlapped filters must let through no light at all so that a bright light source is hardly visible.
4. Set the Contrast of the monitor to the maximum value of 100.
5. Set the Brightness of the monitor as low as possible with such a goal that the black on the monitor look absolutely black (rather than pale gray). With modern high luminance monitors your brightness values may happen to be as low as 10.

In a case of a screen projector...

- The light in the room must be zero.
- Set the Contrast of the projector to the maximum value of 100.
- Set the Brightness of the projector as low as possible with such a goal that the black on the screen look absolutely black (rather than pale gray). Your brightness values may happen to be as low as 10.

Both for a desktop and a screen projector the goal of the best setting is such, that there be no ghost images visible, i.e. that the right eye see only the right image in blue and nothing in red, while the left eye see only the left image in red and nothing blue.