The Conjecture

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Content

Introduction and definitions.

This is a review of the surroundings of the Conjecture – a gap in the Unifying view on ODEs, AD and elementary functions [1]. This review is for those wishing to immerse in and solve this problem.

In order to better reveal parallels and relations between various ideas, this report contains several tables and diagrams emphasizing those relations - as an homage to my late mother Ida Levi, a Physics teacher from God, striving to perfection in every of her lectures and presentations, often resorting to tables.

Conventions

- All along this review we are speaking about holomorphic functions and their derivatives in the complex space. We use the acronyms ODEs for Ordinary Differential Equations, and IVP for Initial Value Problem.

- We are to deal with rational and polynomial *systems* of several first order ODEs, and with *stand alone* n-order ODEs.

- We distinguish *explicit* ODEs such as

$$\begin{aligned} x^{(n)} &= F(t,\ x, ...,\ x^{(n-1)}) \quad \text{or} \\ y' &= G(t,\ x,\ y,\ z, ...) \end{aligned}$$

  from *implicit* ODEs

$$P(t,\ x, ...,\ x^{(n)}) = 0$$

- Speaking about explicit rational ODEs

$$\begin{aligned} x' &= \frac{g(t,\ x,\ y,\ z, ...)}{h(t,\ x,\ y,\ z, ...)} \quad \text{or} \\ x^{(n)} &= \frac{P(t,\ x, ...,\ x^{(n-1)})}{Q(t,\ x, ...,\ x^{(n-1)})} \end{aligned} \tag{1}$$

  we presume that no denominator is identical zero, i.e. h or Q may disappear only at a subset of the phase space. We understand that rational ODEs (1) are equivalent to implicit polynomial ODEs

$$\begin{aligned} x'h(t,\ x,\ y,\ z, ...) - g(t,\ x,\ y,\ z, ...) &= 0 \quad \text{or} \\ x^{(n)}Q(t,\ x, ...,\ x^{(n-1)}) - P(t,\ x, ...,\ x^{(n-1)}) &= 0 \end{aligned} \tag{2}$$

  which, unlike the rational (1), are defined at all points of the phase space $(t,\ x,\ y,\ z, ...) \in \mathbf{C}^{m+1}$ or $(t,\ x, ...,\ x^{(n-1)}) \in \mathbf{C}^{n+1}$.

- If the denominator $h(t_0,\ x_0,\ y_0,\ z_0, ...) = 0$ in (1), this explicit rational ODE is called singular at a point $(t_0,\ x_0,\ y_0,\ z_0, ...)$, whose meaning requires a special definition (below).

- Unlike a rational ODE (1), an implicit ODE $P(t,\ x, ...,\ x^{(n)}) = 0$ is defined at all points of its phase space, but it is called singular at a point $(t_0,\ x_0,\ x_1, ...,\ x_n)$ if $\left.\frac{\partial P}{\partial X_n}\right|_{(t_0,\ x_0,\ x_1, ...,\ x_n)} = 0 \quad (\ X_i = x^{(i)})$.

- When we say that a *holomorphic* solution $x(t)$ satisfies a singular at t_0 rational ODEs (1), this means that $x(t)$ satisfies the respective implicit polynomial ODE (2) (while the indefiniteness $0/0$ in the right hand side of (1) may be resolved via obtaining the limit).

These are the conventions for the rest of this report.

The Conjecture

In either of the two equivalent forms below, **the Conjecture** claims that...

<table>
<tr><td colspan="2">For any component, say $x(t)$, of an explicit system in $x(t),\ y(t),\ z(t), ...$ of m 1st order ...</td></tr>
<tr><td>... rational ODEs

$$\begin{aligned} x' &= \frac{g_1(t,\ x,\ y,\ z, ...)}{h_1(t,\ x,\ y,\ z, ...)} \\ y' &= \frac{g_2(t,\ x,\ y,\ z, ...)}{h_2(t,\ x,\ y,\ z, ...)} \\ z' &= \frac{g_3(t,\ x,\ y,\ z, ...)}{h_3(t,\ x,\ y,\ z, ...)} \\ &\quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned} \tag{3}$$

regular at the initial point $(t_0,\ a,\ b,\ c, ...)$, ie. all the denominators $h_k|_{t=t_0} \neq 0$ so that the solution $x(t)$ exists having all the derivatives $x^{(i)}|_{t=t_0} = a_i,\ i = 0,\ 1,\ 2, ...,$</td>
<td>... polynomial ODEs

$$\begin{aligned} x' &= p_1(t,\ x,\ y,\ z, ...) \\ y' &= p_2(t,\ x,\ y,\ z, ...) \\ z' &= p_3(t,\ x,\ y,\ z, ...) \\ &\quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned} \tag{4}$$

regular at all points $(t_0,\ a,\ b,\ c, ...)$ of its phase space so that at any initial point t_0 the solution $x(t)$ exists having all the derivatives $x^{(i)}|_{t=t_0} = a_i,\ i = 0,\ 1,\ 2, ...,$</td></tr>
<tr><td colspan="2">there exists an explicit n-order rational ODE

$$x^{(n)} = \frac{F(t,\ x,\ x', ..., x^{(n-1)})}{G(t,\ x,\ x', ..., x^{(n-1)})} \tag{5}$$

or an implicit polynomial ODE

$$Q(t,\ x,\ x', ..., x^{(n)}) = 0 \tag{6}$$

satisfied by $x(t),\ x^{(i)}|_{t=t_0} = a_i,\ i = 0,\ 1,\ 2, ...,$ both (5) and (6) being regular at the initial point $(t_0,\ a_0, ..., a_n)$.</td></tr>
</table>

As of the moment (October, 2021), this Conjecture is neither proved nor disproved, though a few weaker versions of it were proved.

For example, if the Conjecture is simplified dropping the requirement of regularity in ODEs (5) or (6), the so modified version is true and proved (Appendix 1).

The Theorem in Appendix 1 proves the Conjecture excluding some subset of the initial values. Namely, the claim of the Conjecture for the polynomial ODEs (4) applies to the initial points in the entire phase space $\mathbf{C}^{m+1} = \{(t_0,\ a,\ b,\ c,...)\}$, but the Theorem is proved for $\mathbf{C}^{m+1}\backslash\mathcal{F}$, where

$$\mathcal{F} = \{(t_0,\ a,\ b,\ c,...) \mid q(t_0,\ a,\ b,\ c,...) = 0\}$$

is a manifold, q being some incidental polynomial emerging in the method of the proof.

Definitions of elementariness.

Vector elementariness	*Scalar elementariness*
Definition 1. A function $x(t)$ (as a part of a vector $(x,\ y,\ z,...)$) is called vector-elementary at and near a point $t = t_0$, if it satisfies systems (3) or (4).	**Definition 2**. A function $x(t)$ is called scalar-elementary at and near a point $t = t_0$, if it satisfies an ODE (5) or (6).
Definition 3. A holomorphic at $t = t_0$ vector-function $(x,\ y,\ z,...)$ is called *non-elementary, or violating, or losing its vector elementariness* at $t = t_0$ if in its neighborhood (excluding $t = t_0$ itself) vector-function $(x,\ y,\ z,...)$ can satisfy rational system (3) only if (3) is singular at $t = t_0$. A vector-function which cannot satisfy any rational system (3) at all is called non-elementary everywhere. A *singular* at $t = t_0$ vector-function $(x,\ y,\ z,...)$ is considered *non-elementary* at $t = t_0$.	**Definition 4**. A holomorphic at $t = t_0$ function $x(t)$ is called non-elementary, or violating, or losing its scalar elementariness at $t = t_0$ if in its neighborhood (excluding $t = t_0$ itself) function $x(t)$ can satisfy rational ODE (3) only if (3) is singular at $t = t_0$. A function which cannot satisfy any rational ODE (3) is called non-elementary everywhere. A *singular* at $t = t_0$ function $x(t)$ is considered *non-elementary* at $t = t_0$.

Functions whose scalar elementariness is violated at isolated points do exist and are discussed further along this report.

Remark 1 *Elementariness of a function $x(t)$ at some point t_0 does not mean as though arbitrarily chosen system (3) satisfied by $x(t)$ is necessarily regular at t_0. As the Appendix 4 demonstrates, any ODE or their system may be intentionally made singular at any point. That is why an* arbitrarily *chosen ODEs satisfied by $x(t)$ may happen to be singular despite elementariness of $x(t)$ at the point. For example elementary function $x(t) = t^n$ at $t = 0$ satisfies both regular ODE $x' = nt^{n-1}$ and singular ODE $x' = \frac{nx}{t}$.*

Remark 2 *In order to establish elementariness of a function $x(t)$ at a point t_0 (in either sense) it's necessary to produce a system (3) satisfied by $x(t)$ and* regular *at t_0. From t_0, by integrating the ODEs, a property of elementariness may be analytically continued towards any point t where the denominators h_k or G remain nonzero. On the contrary...*

Remark 3 *In order to establish scalar non-elementariness of a function $x(t)$ at a point t_0 it's not sufficient to merely produce an ODE (3) satisfied by $x(t)$ and singular at t_0: the ODE with a singularity at t_0 may happen to be replaceable with a regular one. It's a challenge to prove non-elementariness of a function. Besides the Euler's Gamma function (non-elementary in either sense at all points), among other functions, so far only scalar non-elementariness was discovered for a special kind of functions (discussed below).*

As any n-order ODE (5) is trivially transformable into a system of n first order ODEs (3), the following relationships between these definitions above take place:

- *Vector elementariness* follows from *scalar elementariness*;
- *Scalar non-elementariness* follows from *vector non-elementariness.*

However it is not known ...

- Whether *scalar elementariness* follows from *vector elementariness*;
- Whether *vector non-elementariness* follows from *scalar non-elementariness.*

From rational to polynomial systems of ODEs.

- **At a regular point**. As it was shown in [1], at any point $t = t_0$ of its regularity a rational ODE or a rational system of ODEs (3) may be converted into a wider system of *polynomial* ODEs (4) in the right column of the Conjecture

$$\begin{array}{l} x' = P(t,\ x,\ y,\ z, ...) \\ y' = Q(t,\ x,\ y,\ z, ...) \\ \quad ...., \end{array} \tag{7}$$

or even further into the special polynomial systems in Table 2 (5-7). Therefore the rational regular system (3) in the Conjecture (and in the definition of *vector elementariness*) at a regular point may be replaced with the polynomial system (7) and special polynomial systems thanks to the Fundamental transforms (Table 2).

- **At a singular point**. Suppose a holomorphic function $x(t)$ satisfies a rational ODE (5) singular at $t = t_0$, and t_0 really is a point of violation of the scalar elementariness of $x(t)$ so that it's impossible to find a regular rational ODE satisfied by $x(t)$ at $t = t_0$, $x^{(k)}|_{t=t_0} = a_k$, where $G(t_0,\ a_0,\ ...,\ a_{n-1}) = 0$. In this scenario we also can transform ODE (5) into a poly system at any point $(t_1,\ b_0,\ ...,\ b_{n-1})$ where $G(t_1,\ b_0,\ ...,\ b_{n-1}) \neq 0$. Consider some point $(t_1,\ b_0,\ ...,\ b_{n-1})$, t_1 near t_0, $t_1 \neq t_0$, $b_k = x^{(k)}|_{t=t_1}$ such that $G(t_1,\ b_0,\ ...,\ b_{n-1}) \neq 0$. A solution of this regular IVP is the same $x(t)$. Now we can transform (5) as in the regular case. Introduce a new variable $y = \dfrac{1}{G(t,\ x,\ x', ..., x^{(n-1)})}$. Then

$$\begin{array}{rcl} x^{(n)} & = & yF(t,\ x,\ x', ...,\ x^{(n-1)}), \qquad x^{(k)}|_{t=t_0} = a_k, \quad k = 0,\ 1, ... \\ y' & = & -y^2 \dfrac{dG}{dt} \end{array} \tag{8}$$

which can be further transformed into the format of the system (4) of first order poly ODEs. We obtained a poly system (8) having the solution $x(t)$, but it's easy to show that while integration of the IVP for $x(t)$ in the system (8), the point $(t_0,\ a_0,\ ...,\ a_{n-1})$ is unreachable because of singularity of y at $t = t_0$.

If we succeeded to obtain a poly system (8) for $x(t)$ in which the point $(t_0,\ a_0,\ ...,\ a_{n-1})$ were reachable, that would be a counterexample disproving the Conjecture. However, if a counterexample exists, the method of transformation above does not deliver it because the point $t = t_0$ happens to be unreachable in specifically *this system* (8) obtained via specifically *this* known method of transformation. If we proved that the point $t = t_0$ is unreachable in any poly system (4) satisfied by $x(t)$, that would be a proof of the Conjecture. The following Claim therefore is ...

An alternative form of the Conjecture

Claim 1 *Let a holomorphic point t_0 in a function $x(t)$ be a point of violation of scalar elementariness. Then in any poly system (7) satisfied by $x(t)$ at least one of the components $y,\ z, ...$ of (7) must be singular at the point t_0 making t_0 unreachable during integration of the system (7) of ODEs.*

Explicit polynomial systems of ODEs (7) do not have singular points in their phase space (though their solution vectors may have singularities). This fact has an interesting implication.

Remark 4 *The concept of* vector elementariness *of a function $x(t)$ at a point is based on* regularity *of ODEs at the respective point of their phase space. That is why the fact of elementariness may be expressed either via* rational *(3) or* polynomial *ODEs (7). However the opposite concept of* non-elementariness *of a function $x(t)$ is based on singularity of ODEs at the respective point of their phase space. Therefore* non-elementariness *may be expressed only via* rational *ODEs (3) where the denominator disappears at some points of the phase space.*

Evolution of the concept of elementary functions

The predecessor of this definition of the *vector-elementariness* was the definition by Ramon Moore in the 1960s. However Moore's definition was not linked to a point, nor did it require regularity of the right hand sides of the rational ODEs. Moore's definition defined elementariness for a vector-function in its entire domain of existence.

The classical (conventional) definition of elementary functions in the 19th century by Liouville also applied to the entire domain of existence of the functions. It's easy to demonstrate that all functions elementary by Liouville are also elementary by Moore.

We ought to **refine** the concept of elementariness making it specific to a neighborhood of a point **because of the new results** [2], namely, that *scalar elementariness* may be lost at isolated points in some functions like $\dfrac{e^t-1}{t}$ or $\dfrac{\sin t}{t}$ at $t=0$ (by Liouville and Moore they are merely elementary everywhere). The loss of elementariness at a point is proved only for scalar elementariness, and it is not known whether these functions lose also *vector elementariness* at the same (or other) point – an open question depending on the Conjecture.

And vice versa, there exist other properties for which the proofs are known only for *vector-elementariness* [1]. They are...

- Closeness of the class of vector-elementary functions in respect to the superposition and the inverse vector-functions, and ...
- The Fundamental transforms – see Table 2.

We will discuss the loss of elementariness at a point later in this review.

It's worth noting that functions may be non-elementary also at *all points* of their domain by any of the above definitions. At the moment the only known example of a function non-elementary at all points is the Euler's Gamma function and Gamma integral [1], whose non-elementariness follows from the Hölder theorem about the Gamma function.

The following table summarizes the facts presented above:

Scalar elementariness	**Vector elementariness**
	Follows from the scalar elementariness, but the vice versa statement is not yet proved.
The proofs of the fundamental properties *not known*	The fundamental properties **proved**: the closeness of the class of vector-elementary functions in respect to the superposition and the inverse vector-functions, and the fundamental transforms (see the Table of transforms)
Not known if contains all Liouville elementary functions	**Contains** the conventional Liouville elementary functions
Is **lost** at isolated points in some functions like $\frac{e^t - 1}{t}$ or $\frac{\sin t}{t}$ at $t = 0$.	*Not known* if it is lost at these points
The Γ function and Γ integral are non-elementary at all points in any sense thanks to the Hölder theorem.	

Table 1. Comparison of the facts which follow from each of the two definitions

<table>
<tr><td colspan="3">(1) An explicit first order system of ODEs whose right-hand side is a vector-elementary function converts to...</td></tr>
<tr><td colspan="3">(2) A system of ODEs whose right-hand sides are rational functions. At regular points it further converts to...</td></tr>
<tr><td rowspan="3">(3) A canonical system: an explicit system of algebraic and differential equations for computing n-order derivatives requiring $O(n^2)$ operations.</td><td colspan="2">(4) A system, whose right-hand sides are polynomials. It further converts to...</td></tr>
<tr><td colspan="2">(5) Polynomial ODEs of degree $\leq$ 2. It further converts to polynomial ODEs of degree 2 with ...</td></tr>
<tr><td>(6) ...with coefficients 0, 1 only (Kerner)</td><td>(7) ... with squares only (Charnyi)</td></tr>
</table>

Table 2. Fundamental transforms

The Conjecture: discussion

The Conjecture is not merely an isolated open question in itself, but it represents a gap in the Unifying View [1].

The Conjecture claims that the scalar elementariness at a point does follow from the vector elementariness at the same point. The vice versa statement is trivial, therefore the Conjecture (if true) establishes *equivalence* between the *vector* and *scalar* elementariness.

It will be further proven (Appendix 1) that conversion from the rational (3) or poly system (4, 7) to an implicit polynomial ODE (6) or a rational ODE (5) (singular or regular at t_0) is always possible, so that it's a requirement of regularity of the ODE (5) or (6) which is a stumbling block.

Remark 5 *The phase space of initial points* $(t_0,\ a_0,\ b_0, ...)$ *of the poly system (7) is* $\mathbf{C}^{m+1}$*, and the phase space of points* $(t_0,\ a_0,\ a_1, ...)$ *of rational ODE 5) is* $\mathbf{C}^{n+2}$*. The function* $x(t)$ *in the Conjecture therefore establishes a mapping* $(t_0,\ a_0,\ b_0, ...) \longmapsto (t_0,\ a_0,\ a_1, ...)$

There are the following options for a function $x(t)$ within the setting of the Conjecture.

- Either the function $x(t)$ does not have any points of violation of its scalar elementariness at all, so that for any given point there must exist a regular ODE (5) satisfied by $x(t)$ (no matter whether we can produce such an ODE). For this case the Conjecture therefore is true.

- Or, being scalar elementary at t_0, the solution $x(t)$ does have some point $t_1 \neq t_0$ of violation of its scalar elementariness. Then in order to prove the Conjecture we must produce a regular at t_0 rational ODE (5) (or prove its existence without production). What the Conjecture means in this case is that the point of non-elementariness $(t_1,\ a_0,\ a_1, ...)$ of the phase space of (5) is an image of no point $(t_0,\ a_0,\ b_0, ...)$ of the phase space of the polysystem (7): otherwise the polysystem (7) and $x(t)$ would be a counterexample demonstrating that the Conjecture is false.

- Or t_0 is a point of violation of scalar elementariness of $x(t)$ so that the polysystem (7) and $x(t)$ would be a counterexample demonstrating that the Conjecture is false.

The following **gap diagram** shows the place of the Conjecture within the general problem of transforming a system of ODEs into one ODE and vice versa.

Given...

a system of m explicit first order ODEs at a regular point $\left\{\begin{array}{l} \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ u_k' = g_k(t,\ u_1, ...,\ u_m) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}\right.$	vs.	one n-order ODE in u_1 at a regular point $u_1^{(n)} = f(t,\ u_1, ...,\ u_1^{(n-1)})$

the following transformations "→"take place:

Source	*Target*	*The target is…*	
		Rational	Holomorphic
One n-order ODE →	System of m 1st order ODEs	Yes	Yes
System of m 1st order ODEs →	One *regular* n-order ODE	?	Yes
	One possibly *singular* n-order ODE	Yes	

where "One possibly singular n-order ODE" means that the target ODE may be either regular or singular (its regularity is not guaranteed). The yellow gap with a question mark stands for the Conjecture.

A possibility of the conversion from a "System of m 1st order ODEs" into "One (possibly singular) n-order ODE" is proved in the Theorem (Appendix 1)

Remark 6 *As the diagram shows, if we do not ask for the rational right hand side in the ODE (5) admitting an* arbitrary holomorphic *right hand side instead, transformation of an explicit system (3) into one explicit ODE (of first order at that!) would be always possible, albeit in a trivial tautological sense. Namely, consider the holomorphic solution $x(t)$ of system (3), and denote $\varphi(t) = x'(t)$. Then the ODE $x' = \varphi(t)$ is the required holomorphic ODE. Moreover...*

Remark 7 *Due to the properties of holomorphic functions, the singular points of this very special right hand side $\varphi(t)$ of this ODE (whose phase space is merely (t)) are the same as the singular points of its solution $x(t)$. Generally, however, the phase space $(t,\ a_{0,}\ a_1, ...)$ of ODE (5) is of a higher dimension, and its set of singular points may differ from the set of singular points of the solution $x(t)$. In particular, polynomial right hand sides (7) have no singular points, "hiding" all possible singular points of the solution $x(t)$. The polynomial right hand sides may hide also singular points of right hand sides of a rational system (3) which was transformed to a wider polynomial system (7).*

Points where elementariness is violated

In the paper [2] a new type of special points in holomorphic functions was discovered: the points where functions lose their property of being *scalar elementary* while being holomorphic. In particular the function

$$x(t) = \frac{e^t - 1}{t}, \quad x^{(k)}|_{t=0} = \frac{1}{k+1}, \quad k = 0,\ 1, ..., \tag{9}$$

the solution of the ODE

$$tx' - tx + x - 1 = 0, \quad \text{or} \quad x' = x - \frac{x-1}{t}.$$

At the point $t = 0$ $x(t)$ can satisfy no explicit rational ODE regular at this point, nor can it satisfy any explicit polynomial ODE

$$x^{(n+1)} = P(t,\ x,\ x',...,\ x^{(n)}) \tag{10}$$

indeed. The same is true for infinitely many other functions[1] at $t = 0$ such as

$$\frac{\sin t}{t},\quad \cos\sqrt{t},\quad \frac{\log(1+t)}{t} \tag{11}$$

as well as for the solution $x(t)$ of the IVP $tx'' - x = 0,\ \ x|_{t=0} = 0,\ \ x'|_{t=0} = 1.$

The existing proof of violation of scalar elementariness capitalizes on a specific general pattern of the values of the derivatives of such functions – see the Corollary 2 in [2]. Those values are irreducible rational numbers with denominators passing through all prime numbers.

Remark 8 *Most of the examples of the points of violation of scalar elementariness here look like indefiniteness in a fraction*[2]*. However not any irreducible fraction with an indefiniteness presents a violation of elementariness. For example the function*

$$x(t) = \frac{+\sqrt{1+t}-1}{t},\quad x(0) = \frac{1}{2} \tag{12}$$

does not violate elementariness at $t = 0$ *because it satisfies an ODE*

$$x' = \frac{x^2}{2tx+2},\quad x(0) = \frac{1}{2}.$$

regular at $t = 0$*. The indefiniteness in the fraction (12) is a consequence that* $x(t)$ *satisfies an algebraic equation* $tx^2 + 2x - 1 = 0$*, and* t *passes through zero in the leading monomial. Violation of scalar elementariness*

[1] All the functions (11) lose their *scalar* elementariness at $t = 0$, however it is not known if they lose their *vector* elementariness at this point. They would, if the Conjecture is true so that the equivalence between the scalar and vector elementariness takes place.

[2] Speaking about an irreducible rational function $\frac{f(x,\ y,\ z,...)}{g(x,\ y,\ z,...)}$, it is elementary at all point where $g(x,\ y,\ z,...) \neq 0$. At the points where $g(x,\ y,\ z,...) = 0$ but $f(x,\ y,\ z,...) \neq 0$ it is surely singular. For multivariate irreducible function f/g it is possible that both $f(x,\ y,\ z,...) = g(x,\ y,\ z,...) = 0$ making an indefiniteness. However it is provable, that multivariate fractions f/g are singular also at the points of indefiniteness. By Definition, the concept of elementariness applies only at regular points of (vector-) functions.

happens not necessarily in fractions. The function $x(t) = \cos\sqrt{t}$ *(courtesy of George Bergman), being not a fraction, also demonstrates violation of elementariness: at the singular point of* $\sqrt{t}$ *- while* $x(t)$ *is holomorphic at* $t = 0$*. However, for most of the functions following the pattern in [2] no finite expressions or the ODEs are known. It's worth noting that the two other examples in (11) represent the remarkable limits, the third one being more recognizable in the form* $u(t) = (1+t)^{1/t}$, $u(0) = \lim_{t\to 0}(1+t)^{1/t} = e$*. However the sequence* $u^{(k)}(0)$, $k = 0, 1, 2, \ldots$ *(unlike that for* $\ln u(t)$ *at* $t = 0$*) does not follow the pattern in [2] so that it is not known if* $u(t)$ *loses elementariness at* $t = 0$*. More examples of functions with points being suspects for loss of scalar elementariness are in [4, 5].*

Except at point $t = 0$, all these functions satisfy rational ODEs, and it was proven that any such rational ODE must be singular at point $t = 0$ meaning that none of them may be an explicit polynomial ODE (10).

Corollary 1 *If a function* $x(t)$ *has a point of violation of its scalar elementariness,* $x(t)$ *can satisfy* no explicit polynomial *ODE (10).*

The equivalent form is...

Corollary 2 *If a function* $x(t)$ *satisfies an* explicit polynomial *ODE (10),* $x(t)$ *is scalar elementary in its entire domain of existence (i.e. it has no points of violation of its scalar elementariness).*

However the vice versa statement is open.

Proposition 1 *(an open statement) If a function* $x(t)$ *is scalar elementary in its entire domain, it must satisfy some explicit* polynomial *ODE (10) rather than a rational ODE (5) according to the Definition.*

If a functions $x(t)$ is scalar elementary in the *entire domain* of its existence, it means that the denominator $g(t,\ x, \ldots,\ x^{(n-1)})$ in its rational ODE (5) must never disappear on $x(t)$ in its entire domain. This seemingly suggests as though the denominator must be a nonzero constant – as the image of complex functions usually is the entire complex space (unless the function is a constant), so that the Proposition seems true. However :

1. Either the denominator g of the rational ODE (5) actually is constant 1 so that the ODE is polynomial – as the Proposition claims;

2. Or the function $g(t,\ x, ...,\ x^{(n-1)})$ varies but never reaches zero on $x(t)$ because zero happened to be the excluded (lacunary) value of the complex function g (in accordance with the Picard theorem) – see the Example below. It's because of the case (2) that the Proposition remains an open statement.

Example 1 *(illustrating case (2)) Observe that for the function* $x(t) \equiv x' \equiv x'' = e^t$ *the lacunary value is zero and* $x(t)$ *satisfies the following ODE*

$$x'' = \frac{(x')^2}{x}, \quad x(0) = x'(0) = 1.$$

Its denominator never disappears on $x(t)$ *thus this ODE never turns singular. Though in this example we do know the alternative polynomial ODE* $x' = x$ *satisfied by* $x(t)$ *(as the Proposition claims), for an arbitrary* $x(t)$ *scalar elementary in its entire domain we do not know whether the Proposition is true.*

In accordance with Corollary 1, a function losing its scalar elementariness at some point cannot satisfy any explicit poly ODE (10) at all. However it can satisfy a *system* of explicit poly ODEs (7). (Any system of explicit poly ODEs "hides" both the points of singularity of the solution $x(t)$, and the points where the denominator of the ODE disappears, because such points happen to be unreachable in a polynomial system - see the following example).

Example 2 *The function* $x(t) = \dfrac{e^t - 1}{t}$, *losing its scalar elementariness at* $t = 0$, *cannot satisfy an explicit polynomial ODE (10), however it can satisfy a* system *of first order poly ODEs. Just introduce* $y(t) = \dfrac{1}{t}$, $y' = -y^2$, *and then the* $x(t)$ *satisfies an IVP for a polynomial system (say at an initial point* $t = 1$*)*

$$\begin{aligned} x' &= x - xy + y, \qquad && x|_{t=1} = e - 1 \\ y' &= -y^2, && y|_{t=1} = 1. \end{aligned}$$

Observe: while the stand-alone function $x(t)$ *as a holomorphic function may be analytically continued into the point* $t = 0$ *via its Taylor expansions,* $x(t)$ *cannot be continued into this point via integration of this IVP because of a singularity of* $y(t)$ *at the point* $t = 0$ unreachable *for* $y(t)$ *– and therefore* *unreachable for the entire system. Also observe, that if the Conjecture claimed that the target ODE were* explicit polynomial *rather than* explicit rational *(5), the Conjecture would be false, as this Example demonstrates: here $x(t)$ is the solution of a polynomial system of ODEs, but it cannot be a solution of one explicit polynomial ODE (10).*

The infinite fundamental sequence of polynomial equations

The polynomial system (7) allows obtaining the infinite sequence of polynomial equations (the so called Fundamental Sequence)

$$\begin{array}{|l|l|l|l|}\hline x' = P_1(t,\ x,\ y,\ z,...) & y' = Q_1(t,\ x,\ y,\ z,...) & z' = ... & ... \\ \hline ... & & & \\ x^{(k)} = P_k(t,\ x,\ y,\ z,...) & & & \\ x^{(k+1)} = P_{k+1}(t,\ x,\ y,\ z,...) & & & \\ ... & & & \\ \hline \end{array} \tag{13}$$

where the following recursive relations take place[3]:

$$P_{k+1}(t,\ x,\ y,\ z,...) = \frac{\partial P_k}{\partial t} + \frac{\partial P_k}{\partial x}x' + \frac{\partial P_k}{\partial y}y' + ... = \frac{\partial P_k}{\partial t} + \frac{\partial P_k}{\partial x}P_1 + \frac{\partial P_k}{\partial y}Q_1 + ...$$

(and the similar infinite sequence may be written also for $y^{(k)},\ z^{(k)},...$ if we needed it).

Remark 9 *According to these recurrent relations, generally the degrees of polynomials P_k grow with k (unless all the right hand sides $P_1,\ Q_1,...$ of (7) are polynomials of a degree ≤ 1). In particular, if the right hand sides of (7) are polynomials of degree 2, the degrees of P_k increase by ≤ 1 because the operators $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, ...$ reduce the degree of the polynomials by 1, while the factors P_1 , Q_1 possibly increase it by 2 so that the degrees of P_k increase by a value ≤ 1.*

Remark 10 *For the further considerations, we can assume that there are infinite number of nonzero values among $\{x^{(k)}|_{t=t_0}\}$ and therefore no P_k is a zero polynomial. If there are only finite number of nonzero values among $\{x^{(k)}|_{t=t_0}\}$, the function $x(t)$ must be a polynomial so that the Conjecture is obviously true in such a case.*

[3] For the polynomial system in squares only the recurrent relations are simpler - see Appendix 5.

Remark 11 *The relation (13) may be viewed...*

- either as ODEs $x^{(k)}(t) = P_k(t,\ x(t),\ y(t),\ z(t), ...)$ where the variables of the polynomials P_k are not independent, being specific functions - the solutions of the system (7);
- or as relations $x_0^{(k)} = P_k(t_0,\ x_0,\ y_0,\ z_0, ...)$ where the variables of the polynomials P_k are independent variables - the points of the phase space of the system.

Remark 12 *The fundamental sequence (13)* $\mathbf{P}_n = (P_1,\ P_2, ...,\ P_n)$ *maps the phase space* $(t_0,\ a_0,\ b_0,\ c_0, ...)$ *of the system (7) onto the phase space* $(t_0,\ a_0,\ a_1,\ a_2, ...,\ a_n)$ *of the ODE (5):*

$$(t_0,\ a_0,\ b_0,\ c_0, ...) \xmapsto{\mathbf{P}_n} (t_0,\ a_0,\ a_1,\ a_2, ...,\ a_n),$$
$$\mathbf{P}_n \colon \mathbf{C}^{m+1} \to \mathbf{C}^{n+2}.$$

At that, the general solutions $x(t;\ t_0,\ a_0,\ b_0,\ c_0, ...)$ *and* $x(t;\ t_0,\ a_0,\ a_1,\ a_2, ...)$ *are different multivariate functions, but for the specific solutions* $x(t)|_{(t_0,\ a_0,\ a_1,\ a_2, ...)} = x(t)|_{(t_0,\ a_0,\ b_0,\ c_0, ...)}$.

Remark 13 *The source of the mapping is the phase space* $\mathbf{C}^{m+1} = \{(t_0,\ a_0,\ b_0,\ c_0, ...)\}$ *of the poly system (7). If we fix certain n, the fundamental sequence (13) maps* $\mathbf{C}^{m+1} \overset{\mathbf{P}_n}{\to} \{(t_0,\ a_0,\ a_1,\ a_2, ...,\ a_n)\}$ *and the image* $I_n = \mathbf{P}_n(\mathbf{C}^{m+1})$ *is some* $m+1$ *dimensional manyfold:* $I_n \subset \mathbf{C}^{n+2}$*, but* $I_n \neq \mathbf{C}^{n+2}$ *so that* $\mathbf{C}^{n+2} \backslash I_n$ *is not empty. Therefore for the target ODE (5) some vectors of initial values* $(t_0,\ a_0,\ a_1,\ a_2, ...,\ a_n) \in \mathbf{C}^{n+2} \backslash I_n$ *correspond to no initial values* $(t_0,\ a_0,\ b_0,\ c_0, ...)$ *of the system (7). Or in other words, some particular solutions* $x(t)|_{(t_0,\ a_0,\ a_1,\ a_2, ...)}$ *of the target ODE (5) match no solution* $x(t)|_{(t_0,\ a_0,\ b_0,\ c_0, ...)}$ *of the source system (7). In particular, if* $(t_0,\ a_0,\ a_1,\ a_2, ...,\ a_n)$ *is a point of violation of scalar elementariness and the Conjecture is true, the point* $(t_0,\ a_0,\ a_1,\ a_2, ...,\ a_n)$ *corresponds to no initial values of the system (7).*

As shown in Appendix 1, a rational ODE (5), regular or singular, may be always obtained from the fundamental sequence. Then, utilizing the expressions for $x^{(k)}$ in the fundamental sequence

$$P_n(t,\ x(t),\ y(t),\ z(t),...) = x^{(n)} = \frac{f(t,\ x,\ x',...,\ x^{(n-1)})}{g(t,\ x,\ x',...,\ x^{(n-1)})} =$$
$$= \frac{f(t,\ x,\ P_1(t,\ x,\ y,\ z,...),...,\ P_{n-1}(t,\ x,\ y,\ z,...))}{g(t,\ x,\ P_1(t,\ x,\ y,\ z,...),...,\ P_{n-1}(t,\ x,\ y,\ z,...))} =$$
$$= \frac{u(t,\ x(t),\ y(t),\ z(t),...)}{v(t,\ x(t),\ y(t),\ z(t),...)}. \tag{14}$$

where u and v are polynomials over dependant variables.

This relation (14) demonstrates the singularities in the target rational ODE (5) either as a subset $g(t_0,\ a_0,...,\ a_{n-1}) = 0$ of its phase space $(t_0,\ a_0,...,\ a_{n-1})$, or as a subset $v(t_0,\ a_0,\ b_0,\ c_0,...) = 0$ of the phase space $(t_0,\ a_0,\ b_0,\ c_0,...)$ of the system (7).

We will see that the target ODE (5) in the Conjecture is not unique, and there are many ways of obtaining it, though neither of the ways we know so far guarantees that the obtained ODE is regular at the given point: otherwise the Conjecture would be proved.

Conclusion 1 *If the Conjecture is true and the target rational ODE (5) was found for the given initial point* $(t_0,\ a_0,\ b_0,\ c_0,...) \overset{\mathbf{P}}{\longmapsto} (t,\ a_0,...,\ a_{n-1})$ *so that at* this *point both* $v(t_0,\ a_0,\ b_0,\ c_0,...) \neq 0$ *and* $g(t,\ a_0,...,\ a_{n-1}) \neq 0$, *generally the polynomial* v *still disappears for a manifold of other points* $(t,\ a,\ b,\ c,...)$ *where* $v(t,\ a,\ b,\ c,...) = 0$. *The target ODE (5) regular at* $(t_0,\ a_0,\ b_0,\ c_0,...)$ *may happen to be singular at other points* $(t,\ a,\ b,\ c,...)$.

Example 3 *(further illustrating how the Conjecture "works"). Consider a polynomial system*

$$\begin{aligned} x' &= x - xy + y \\ y' &= -y^2 \end{aligned} \tag{15}$$

(satisfied by the earlier introduced function (9) $x(t) = \dfrac{e^t - 1}{t}$*). There exist many ways for obtaining the target ODE in* x*. In this special case we can obtain the general solution for* y*. Choose any initial value* $(t_0,\ x_0,\ y_0) \in \mathbf{C}^3$ *of the phase space of the system, and write down the solution component* y *in*

the form $y = \dfrac{y_0}{y_0(t-t_0)+1}$ *for all* y_0. *For* $y_0 \neq 0$ $\quad y = \dfrac{1}{t + \dfrac{1}{y_0} - t_0}$. *Then the corresponding target ODE in* x *containing* $t_0,\ y_0$ *as parameters is*

$$\begin{aligned} x' &= x - \frac{y_0(x-1)}{y_0(t-t_0)+1} \qquad \text{for any } y_0, \text{ or} \qquad (16)\\ x' &= x - \frac{x-1}{t - t_0 + \dfrac{1}{y_0}} = x - \frac{x-1}{t+\delta} \quad \text{for} \quad y_0 \neq 0, \quad \delta = \frac{1}{y_0} - t_0. \end{aligned}$$

The function (9) is defined by a special IVP for the system (15) introduced earlier. Other IVPs of the system (15) defining (9) or its shifted version are explained below. Properties of particular solutions $x(t),\ y(t)$ *belonging to the general solution* $x(t,\ t_0,\ x_0,\ y_0, ...),\ y(t,\ t_0,\ x_0,\ y_0, ...)$ *depend on certain subsets of the phase space.*

Case 1 *Any* $y_0 \neq 0$ *and any* $t_0,\ x_0$ *specifying therefore nonzero solutions* $y(t)$ *(hyperbolas). Denote* $\delta = \dfrac{1}{y_0} - t_0$ *so that* $t_0 + \delta = \dfrac{1}{y_0}$ *and the subset*

$$S_1 = \{(t_0,\ x_0,\ y_0)\ \mid y_0 \neq 0\} \subset \mathbf{C}^3,$$

meaning that $S_1 = \mathbf{C}^3 \backslash \{\text{hyperplane } y_0 = 0\}$. *The ODE (16) takes form*

$$x' = x - \frac{x-1}{t+\delta}$$

regular at $t = t_0$ *but singular at* $t = -\delta$. *At that if* $\lim_{t \to -\delta} x(t) = 1$, *the solution* $x(t)$ *is the* δ*-shifted entire function (9)* $x_\delta(t) = \dfrac{e^{t+\delta}-1}{t+\delta}$, $\ x_\delta(-\delta) = 1$. *The initial values* $(t_0,\ x_0)$ *defining the shifted solutions* $x_\delta(t)$ *are the points* $\left(t_0,\ x_0 = \dfrac{e^{t_0+\delta}-1}{t_0+\delta}\right) = (t_0,\ (e^{1/y_0}-1)y_0)$. *Observe that though* $\lim_{t_0+\delta \to 0} x_0 = \lim_{y_0 \to \infty} x_0 = 1$, *this* x_0 *does not reach* 1 *for whichever* y_0. *Consider the following two subsets of* S_1.

1. *A subset* $S_{11} \subset S_1$, *a subset of points* $(t_0,\ x_0,\ y_0)$ *belonging to the* δ*-shifted curve* $x_\delta(t) = \dfrac{e^{t+\delta}-1}{t+\delta}$

$$S_{11} = \{(t_0,\ x_0,\ y_0) \mid x_0 = (e^{1/y_0}-1)y_0, \quad y_0 \neq 0\}.$$

For initial values $(t_0,\ x_0,\ y_0) \in S_{11}$ *the solution is* $x_\delta(t)$ *whose scalar elementariness is violated at the point* $P_\delta = (t = -\delta,\ x = 1) = (-\delta,\ 1) = \left(t_0 - \dfrac{1}{y_0},\ 1\right)$ *of the phase space of the ODE (16). However the corresponding point* $\left(t_0 - \dfrac{1}{y_0},\ 1,\ y_0\right)$ *of the system (15) does not belong to* S_{11}.

2. *A subset* $S_{12} = S_1 \backslash S_{11}$ *of points* $(t_0,\ x_0,\ y_0)$ *outside of the* δ*-shifted curve* $x_\delta(t) = \dfrac{e^{t+\delta} - 1}{t + \delta}$ *so that* $x_0 \neq (e^{1/y_0} - 1)y_0$. *For the subset* S_{12} *the solution* $x(t)$ *differs from the* δ*-shifted (9) so that the numerator* $(x-1)|_{t=-\delta} \neq 0$ *- therefore the solution* $x(t)$ *must have a singularity at* $t = -\delta$ *(instead of the point of holomorphy whose scalar elementariness is violated). Unlike the case (1), now all solutions for* S_{12} *have a singularity at* $t = -\delta = t_0 - \dfrac{1}{y_0}$.

Case 2 $y_0 = 0$ *and any* t_0, x_0, *meaning the zero solutions* $y(t)$. *The subset*

$$S_0 = \{(t_0,\ x_{0,}\ y_0)\ \mid y_0 = 0\} \subset \mathbf{C}^3$$

is a complex hyperplane in $\mathbf{C}^3$. *Then* $y(t) \equiv 0$ *so that the ODE (16) takes the form* $x' = x$, *whose solution is* $x(t) = x_0 e^{t - t_0}$. *This ODE is regular at all points so that its solutions* $x(t)$ *do not have points of violation of scalar elementariness.*

Conclusion 2 *The phase space of the system (15) is* $\{(t_0,\ x_0,\ y_0)\} = \mathbf{C}^3$, *and the respective solutions are* $x(t;\ t_0,\ x_0,\ y_0)$. *The phase space of the family of ODEs (16) (depending on the parameter* y_0*) is* $\{(t_0,\ x_0)\} = \mathbf{C}^2$ *and the respective solutions are* $x(t;\ t_0,\ x_0;\ y_0)$. *The points of violation of scalar elementariness in solutions of ODEs (16) are* $\left(t_0 - \dfrac{1}{y_0},\ 1,\ y_0\right)$ *but no point* $(t_0,\ x_{0,}\ y_0) \in \mathbf{C}^3$ *maps into them.*

This particular target ODE (16) illustrates the Conjecture in the following way.

- ODEs (16) are rational and *regular* for *any* initial value $(t_0,\ x_{0,}\ y_0) \in \mathbf{C}^3$ of the system (15) as the Conjecture claims.

- As the ODEs (16) corresponding to all point $(t_0,\ x_{0,}\ y_0) \in \mathbf{C}^3$ are regular, it may seem as though $x(t)$ does not have any points of violation of its scalar elementariness, but it does. Violation of scalar elementariness of $x(t)$ takes place at the point $\left(t_0 - \dfrac{1}{y_0},\ 1,\ y_0\right)$ which is an image of no point $(t_0,\ x_{0,}\ y_0)$.

- The family of ODEs (16) also illustrates the fact that different target ODEs (5) correspond to different initial points $(t_0,\ x_{0,}\ y_0)$ of the polysystem (7).

However the target rational ODE (5) could be obtained in another way. As the ODE (16) happened to explicitly depend on the corresponding initial points $t_0,\ y_0$, we can rid of them applying differentiation and finally obtaining the ODE

$$x''(1-x) = x' - 2(x')^2 + 2xx' - x^2.$$

This ODE is remarkable in that it is satisfied by polynomials $x^{(k)} = P_k(t,\ x,\ y),\ \ k = 1,\ 2, ...$ from the fundamental sequence (13). Parameters of the initial vector $(t_0,\ x_{0,}\ y_0) \in \mathbf{C}^3$ do not occur in this ODE so that the same ODE serves as a target ODE for any point $(t_0,\ x_{0,}\ y_0) \in \mathbf{C}^3$. Unlike the ODE (16), this ODE however is singular for the subset of the initial values of (15) where $x_0 = 1$, therefore if instead of the (16) we happened first to obtain this ODEs, it could not illustrate the Conjecture.

Example 4 *Consider the polynomial system*

$$\begin{aligned} x' &= nxy \\ y' &= -y^2 \end{aligned}$$

for natural n*, whose general solution at any initial point* $(t_0,\ x_0,\ y_0)$ *is known:* $x = x_0(y_0(t-t_0)+1)^n, \quad y = \dfrac{y_0}{y_0(t-t_0)+1}$*. Again, an arbitrary process of obtaining the target ODE may provide different yields.*

1. A family of ODEs depending on the parameter y_0

$$x' = \frac{nxy_0}{y_0(t-t_0)+1}.$$

At any initial point $(t_0,\ x_{0,}\ y_0) \in \mathbf{C}^3$ this target ODE is regular;

2. An ODE

$$nx''x = (n-1)(x')^2$$

not specific to initial point $(t_0,\ x_{0,}\ y_0)$ and satisfied even with the polynomial expressions $x^{(k)} = P_k(t,\ x,\ y),\quad k = 1,\ 2, \ldots,$ but singular for the initial values with $x_0 = 0$;

3. An ODE

$$x' = cn(t-t_0)^{n-1}.$$

The first and the third are regular for any $(t_0,\ x_0,\ y_0) \in \mathbf{C}^3$ illustrating the claim of the Conjecture. At that the third is not merely rational, but an explicit polynomial ODE (which is possible because its polynomial solution $x = x_0(y_0(t-t_0)+1)^n$ does not have points where scalar elementariness is violated). However the second one does *not* illustrate the claim of the Conjecture being singular for the initial points of the system where $x_0 = 0$. Not any arbitrarily obtained target ODE (16) illustrates the Conjecture.

As we will see in Appendix 1, the challenge of proving the Conjecture is in that so far the only known way of converting the general polynomial system of ODEs into one ODE yields a target ODE (6) not guaranteeing its regularity $\left.\frac{\partial Q}{\partial X_n}\right|_{t=t_0} \neq 0$.

The Conjecture vs. a problem of regularization of an ODE

We have seen that a function $x(t)$ holomorphic at some point t_0 may satisfy many different ODEs (6): either regular or singular at this point. According to the Theorem in Appendix 1 we can assume that any polynomial system (7) is already transformed into an implicit polynomial ODE (6), regular or singular at a given point t_0. With that in mind, we can pose the following question.

Question 1 *Let a holomorphic function $x(t)$ satisfy an implicit poly ODE*

$$P(t,\ x,\ x',\ \ldots,\ x^{(m)}) = 0$$

which happened to be singular at t_0. Is it always possible to replace the ODE $P = 0$ singular at t_0 with another implicit poly ODE

$$Q(t,\ x,\ x',\ ...,\ x^{(n)}) = 0$$

satisfied by $x(t)$ yet regular *at t_0?*

Sometimes it is possible, but generally the answer is *no*, because some functions like (9) or (11) can satisfy only those implicit poly ODEs (6) which are singular at $t = 0$. The following examples demonstrates it for quite similar ODEs.

Example 5 *The (entire) function $x(t) = te^t$ satisfies the ODE*

$$P = tx' - tx - x = 0$$

singular at $t = 0$. However $x(t)$ satisfies also the ODE

$$Q = x'' - 2x' + x = 0$$

regular at $t = 0$.

Example 6 *The (entire) function $x(t) = \dfrac{e^t - 1}{t}$ satisfies ODE*

$$P = tx' - tx + x - 1 = 0 \tag{17}$$

singular at $t = 0$. However it's proven that there can not exist a polynomial ODE regular at $t = 0$ satisfied by this function. Nor can it satisfy a polynomial ODE with a nonzero constant factor at the leading derivative – because unlike in the Example 5, here function $x(t)$ does have a point of violation of its scalar elementariness.

Example 7 *The (entire) function $x(t) = \dfrac{e^t - 1}{t}$ also satisfies ODE*

$$x''(x-1)(t-1) - (tx' - 2x' + x)(x' - 1) = 0$$

singular at $t = 1$ (with any x, in particular with $x(1) = e - 1$). However, according to Example 6, $x(t)$ satisfies also the ODE (17) regular at $t = 1$ (though with a non-constant *nonzero factor t at the leading derivative). This $x(t)$ however can not satisfy a polynomial ODE with a* constant *nonzero factor at the leading derivative – because unlike in the Example 5, here function $x(t)$ does have a point of violation of its scalar elementariness.*

Which kind of singular ODEs (6) may be replaced (regularized) at a particular point, and which may not? Here is a hypothetical Criterion how this question possibly depends on the Conjecture.

Criterion 1 *If the Conjecture is true, a criterion that an implicit poly ODE (6) satisfied by $x(t)$ and singular at point $t = t_0$ may be replaced with a poly ODE regular at $t = t_0$ is that $x(t)$ at this point satisfies some explicit polynomial system (7).*

If the Conjecture is false...

As we see from the Introduction and the Comparison Table 1, if the Conjecture is false and the competing definitions of elementariness are not equivalent, the properties following from these definitions create a rather complicated picture of the reality (if it is the reality).

If the Conjecture is false, it means that there exists a system of polynomial ODEs (7) (demonstrating vector-elementariness of $(t,\ x(t),\ y(t),\ z(t), ...))$ and such initial values $(t_0,\ x_0,\ y_0,\ z_0, ...)$ of the polysystem (7), that $(t_0,\ x_0,\ x_1, ...)$ (the image of $(t_0,\ x_0,\ y_0,\ z_0, ...))$ is the point of violation of scalar elementariness of the function $x(t)$, i.e. that $x(t)$ can satisfy only singular rational ODEs (5).

In order to find such a counterexample for the Conjecture, it makes sense to consider the examples of functions (9) and (11) for which a loss of their scalar elementariness at $t = 0$ has already been established, and to investigate if they can satisfy some polynomial system (7) at $t = 0$. However this is so far an open question too.

Then, if we find an example of function $x(t)$, which violates the scalar elementariness at point t_0, yet remains vector elementary at the same point, we can pose a question whether the vector elementariness may be violated at certain points.

Remark 14 *If the Conjecture is false so that scalar- and vector-elementariness are not equivalent, we can replace these both definitions with* weak vector- *and* weak scalar-elementariness *respectively by dropping the condition of regularity in the target rational ODEs. The weak vector- and weak scalar-elementariness are equivalent (as proved in Appendix 1) so that we can operate simply with the concept of weak elementariness (both in vector and scalar* *sense). The special points currently defined as the point of violation of the scalar- (and possibly vector-) elementariness would stay in this theory anyway as proven facts.*

Appendix 1: Reduction of a poly system to an implicit ODE

Here we are to prove the "simplified version" of the Conjecture ignoring the requirement of the regularity of the target ODE (6). The proof (whose idea belongs to the late Prof. Harley Flanders[4]) capitalizes on the fact from the combinatorics that the number $\pi(n)$ of partitions of n grows faster than the number of all monomials in r variables of degree $\leq n$ (for any fixed r) – the Lemma 1 in Appendix 2.

Theorem 1 *For every component, say* $x(t)$ *, of a polynomial* system *of ODEs (7) there exists an implicit polynomial ODE (6) the same for all points* $(t_0,\ x_0,\ y_0,\ z_0, ...)$ *satisfied by* $x(t)$*.*

Proof 1 *According to the* Fundamental transforms*, the source rational system at a regular point may be transformed into polynomial systems of various structures including those whose right hand sides are the polynomials of degree 2, and then the degrees of polynomials* P_k *in (13) grow by* 1*. With that in mind, re-write the fundamental sequence (13) in a slightly different notation*

$$\boxed{\begin{array}{ll} x' & = F_2(t,\ x,\ y,\ z, ...) \\ ... & \\ x^{(k)} & = F_{k+1}(t,\ x,\ y,\ z, ...) \\ x^{(k+1)} & = F_{k+2}(t,\ x,\ y,\ z, ...) \\ ... & \end{array}}$$

so that the index k *in polynomials* F_k *stands for its highest degree (in one of its variables). At that, assume that* $F_1 = At + Bx$ *– an arbitrary linear form*[5] *with some constants* $A,\ B$ *(not both zeros).*

Consider a set

$$S_n\ = \{\ (\alpha_1, ...,\ \alpha_n)_i \mid \alpha_1 + 2\alpha_2 + ... + n\alpha_n = n\}$$

of partition vectors $v_i = (\alpha_1, ...,\ \alpha_n)_i,\ i = 1,\ 2,\ 3, ...,\ \pi(n)$ *each representing partitions of* n*, and consider special monomials* $F_1^{\alpha_1} F_2^{\alpha_2} ... F_n^{\alpha_n}$*, keeping in*

[4]From the private communications in 2004

[5]We introduced these free coefficients A and B merely to demonstrate that the proof works with arbitrary A and B.We do not know how to utilize these coefficients for advancing the Conjecture.We can assume as well that $F_1 = x$ o r $F_1 = t$.

mind the bijection

$$(\alpha_1, ..., \alpha_n) \Longleftrightarrow F_1^{\alpha_1} F_2^{\alpha_2} ... F_n^{\alpha_n}.$$

Observe, that after completion of all the operations inside, every monomial $F_1^{\alpha_1} F_2^{\alpha_2} ... F_n^{\alpha_n}$ *becomes an* $n-$*degree polynomial in* $t,\ x,\ y,\ z, ...$ *whose monomials are* $t^\alpha x^\beta y^\gamma z^\delta ...,\ (\alpha+\beta+\gamma+\delta+... \leq n)$. *The number of such monomials* $t^\alpha x^\beta y^\gamma z^\delta ...$ *does not exceed* C_{n+r}^r.

According to the Lemma 1 (Appendix 2), for any given number r *of variables, there exists such a number* n_r *that for any* $n > n_r$ *the number* $\pi(n)$ *of different partitions* $(\alpha_1, ..., \alpha_n)$ *exceeds the number* C_{n+r}^r *of all degree monomials* $\{t^\alpha x^\beta y^\gamma z^\delta ...\}$ *in* r *variables with the degrees* $\leq n$. *Every monomial* $F_1^{\alpha_1} F_2^{\alpha_2} ... F_n^{\alpha_n}$ *is an* $n-$*degree polynomial comprised of the monomials* $\{t^\alpha x^\beta y^\gamma z^\delta ...\}$. *Therefore the set of monomials* $\{F_1^{\alpha_1} F_2^{\alpha_2} ... F_n^{\alpha_n}\}$ *is linearly dependent in* $\{t^\alpha x^\beta y^\gamma z^\delta ...\}$ *so that there exist a linear combination with nonzero coefficients* $a_{\alpha_1, ..., \alpha_n} \neq 0$ *such that*

$$\sum_{(\alpha_1, ..., \alpha_n)} a_{\alpha_1, ...,\ \alpha_n} F_1^{\alpha_1} F_2^{\alpha_2} ... F_n^{\alpha_n} = 0 \tag{18}$$

corresponding to the polynomial ODE

$$U(t,\ x,\ x', ...,\ x^{(n-1)}) = \sum_{(\alpha_1, ...,\ \alpha_n)} a_{\alpha_1, ...,\ \alpha_n} (At + Bx)^{\alpha_1} X_1^{\alpha_2} ... X_{n-1}^{\alpha_n} = 0 \tag{19}$$

where, as usual $X_n = x^{(n)}$.

Impractical just like the method of elimination via the resultants, this Theorem too does not offer a feasible method for obtaining the target ODE. However, unlike the resultants, this Theorem guarantees that the target implicit nonzero polynomial ODE $U = 0$ does exist.

The polynomial U must not necessarily contain all the derivatives X_i up to the highest X_{n-1}. Let k be the highest order of the derivatives X_i in $U : k \leq n-1$. Re-write U by degrees of $X_k :\ X_k,\ X_k^2, ...,\ X_k^\nu$. The degree of the polynomial for X_k is $k+1$, therefore $(k+1)\nu \leq n$ and $\nu \leq \dfrac{n}{k+1}$:

$$U = \sum_{i=0}^{\nu} p_i(t,\ X,\ X_1, ...,\ X_{k-1}) X_k^i = 0 \tag{20}$$

where coefficients $p_i(t,\ X,\ X_1, ...,\ X_{k-1})$ are polynomials.

Remark 15 *The polynomial* U *is satisfied not merely by the solution* $x(t)$, *but by the polynomials* F_k *so that*

$$U(t,\ x,\ F_2(t,\ x,\ y,\ z,...),...,\ F_{n-1}(t,\ x,\ y,\ z,...))$$

is a zero polynomial in $t,\ x,\ y,\ z,...$. *At that the condition of the regularity of* U

$$\frac{\partial U(t,\ X,\ X_1,...,\ X_k)}{\partial X_k} \tag{21}$$

generally does depend on the initial values $t,\ x,\ x',...,\ x^{(k)}$ *and through them depends on the initial point* $(t,\ x,\ y,\ z,...)$ *– unless* $\dfrac{\partial U}{\partial X_k}$ *happened to be a nonzero constant (like in the case* 1 *below).*

Given the properties of the partitions of n, the following outcomes are possible:

1. The highest part of n in the linear combination (18) is n so that there is a monomial with $\alpha_1 = ... = \alpha_{n-1} = 0,\ \alpha_n = 1$, while the highest order k of the derivatives is $n-1$ and this derivative X_{n-1} appears only linearly. Then the polynomial coefficient p_1 at X_{n-1} in (20) must have a degree 0 being reduced to a $const \neq 0$. At that not merely is U linear in X_{n-1}, but the critical value

$$\frac{\partial U(t,\ x,\ x',...,\ x^{(n-1)})}{\partial X_{n-1}} = p_1 = const \neq 0$$

so that the implicit polynomial U turns into an explicit polynomial ODE $x^{(n-1)} = ...$ (10) thus proving a statement even stronger than the Conjecture. As it was noted earlier, this particular case cannot take place *always*, and the Conjecture would be false if it claimed so. When $p_1 = const \neq 0$, the function $x(t)$ does not have the points of violation of scalar elementariness at all (due to the previous Theorems), yet it does not always take place. For example, the function $x(t)$ (9) has the point $t = 0$ where the scalar elementariness is violated, and it satisfies the polynomial system (15) at all points except at $t = 0$. Therefore for the system (15) the outcome can never be this Case 1.

2. The highest partition of n in the linear combination (18) is within $[n/2;\ n-1]$ and respectively $k \in [(n-1)/2;\ n-2]$. Then U is still linear in the highest derivatives X_k however now the critical value

$$\frac{\partial U(t,\ x,\ x',...,\ x^{(k)})}{\partial X_k} = p_1(t,\ X,\ X_1,...,\ X_{k-1})$$

generally is non-constant, and it's not known whether p_1 is nonzero at the initial point. The implicit polynomial U turns into an explicit rational ODE (5) regular or singular at the initial point.

3. The highest partition of n in the linear combination (18) is less than $n/2$ so that the ODE (20) for X_k may be nonlinear. Now in order to obtain an explicit rational ODE (5), it suffices to differentiate (20) applying the operator $\dfrac{d}{dt}$

$$\frac{\partial U}{\partial X_k}x^{(k+1)} + \frac{\partial U}{\partial X_{k-1}}x^{(k)} + ... + \frac{\partial U}{\partial T} = 0.$$

However after differentiation $\dfrac{d}{dt}$, this new ODE may not necessarily be satisfied with polynomials F_k (though we do not know if this fact is helpful for proving the Conjecture). We can avoid this Case 3 at all if we modify the proof of the Theorem choosing n so big that even $\pi(n/2) > C^r_{n+r}$: then $\alpha_n = 1$ so that even the set of monomials $\{F_1^{\alpha_1}F_2^{\alpha_2}...F_n\}$ is linearly dependent in $\{t^\alpha x^\beta y^\gamma z^\delta...\}$.

Conclusion 3 *This theorem proves the Conjecture with the following limitation. The critical factor* $p_1(t,\ t,\ x,\ x',...,\ x^{(k)})$ *may be re-written into a polynomial of* $(t,\ x,\ y,\ z,...)$

$$p_1(t,\ t,\ x,\ x',...,\ x^{(k)}) = q_1(t,\ x,\ y,\ z,...)$$

which generally speaking is not a const. If it happens that $p_1 = const \neq 0$, *the Conjecture is proved. Otherwise* $q_1(t,\ x,\ y,\ z,...)$ *is a nonzero polynomial so that* $q_1(t_0,\ a,\ b,\ c,...) = 0$ *defines a manifold* $\mathcal{F}$ *in the phase space* $(t_0,\ a,\ b,\ c,...)$ *at points of which the critical factor* $p_1 = 0$. *The statement of the Conjecture is made for the entire space* $\mathbf{C}^{m+1}$, *but it is proved here for* $\mathbf{C}^{m+1}\backslash\mathcal{F}$.

Remark 16 *If one ODE (19) is found, infinitely many of them may be obtained by differentiation* $\left(\frac{d}{dt}\right)^N$ *of (19)*

$$U(t,\ x,\ x',...,\ x^{(n-1)})=0$$

$$x^{(n)}\frac{\partial U}{\partial X_{n-1}}+Q_0(t,\ x,\ x',...,\ x^{(n-1)})=0 \tag{22}$$

$$x^{(n+1)}\frac{\partial U}{\partial X_{n-1}}+Q_1(t,\ x,\ x',...,\ x^{(n)})=0$$

$$...$$

$$x^{(n+N)}\frac{\partial U}{\partial X_{n-1}}+Q_N(t,\ x,\ x',...,\ x^{(n+N-1)})=0 \tag{23}$$

$$...$$

where $Q_N \quad (N=0,\ 1,\ ....)$ *denotes some polynomials in the specified variables. Unlike (19), all of these equations are* linear *in the leading derivative* $x^{(n+N)}$ *(the Cases* 1 *and* 2*), but they all have the same critical factor* $\frac{\partial U}{\partial X_{n-1}}=\frac{\partial U(t,\ x,\ x',...,\ x^{(n-1)})}{\partial X_{n-1}}$ *in all equations. Therefore if it happens that* $\left.\frac{\partial U}{\partial X_{n-1}}\right|_{t=t_0}=0$ *in one equation (22) making it singular, all the subsequent ODEs (23) are singular too.*

Remark 17 *This method of proof yields the implicit polynomial ODE universal for all initial points* $(t,\ x,\ y,\ z,...)$*. However, as we had discussed earlier, the target ODE in the Conjecture generally cannot be universal for all points* $(t,\ x,\ y,\ z,...)$*.*

Remark 18 *This method of proof does not take in consideration the specific initial values* $(t,\ x',\ x'',\ ...,\ x^{(k)})$*. It's possible that in some function all these values* $x^{(k)}|_{t=0}=0$ *up to high enough* k *(and nonzero afterwards)*[6]*. As a consequence, the critical factor* $\frac{\partial U(t,\ x,\ x',...,\ x^{(k)})}{\partial X_k}$ *would reduce to* const*, also zero, making (23) singular. Therefore this entire approach which does not take into consideration how many and which* $x^{(k)}|_{t=t_0}$ *are nonzeros cannot lead to the proof of the Conjecture*

[6] If we subtract a segment of the Taylor expansion of a function (i.e. a polynomial) from that function, we can make zero arbitrary many of its beginning derivatives.

Appendix 2 : Boundaries of the partition number

Lemma 1 *The number C^r_{n+r} of all monomials comprising polynomials of degree n in r variables, and the number $\pi(n)$ of partitions of n satisfy the inequalities*

$$C^r_{n+r} < (n+r)^r < 2^{\sqrt{n}} < \pi(n) \tag{24}$$

beginning with a big enough $n > n_r$.

Proof 2 *The inequality $2^{\sqrt{n}} < \pi(n)$ is known say from [3], so we must prove that*

$$(n+r)^r < 2^{\sqrt{n}}, \quad or \quad (n+r)^{\frac{r}{\sqrt{n}}} < 2,$$

which is equivalent to proving that

$$f(n) = \frac{r\ln(n+r)}{\sqrt{n}} < \ln 2.$$

Obtain

$$\begin{aligned} f'(n) &= r\frac{\dfrac{\sqrt{n}}{n+r} - \dfrac{\ln(n+r)}{2\sqrt{n}}}{n} \\ &= r\frac{2n-(n+r)\ln(n+r)}{2(n+r)n\sqrt{n}}. \end{aligned}$$

Beginning from big enough n, a linear function $2n < (n+r)\ln(n+r)$, so that $f'(n) < 0$ and $f(n)$ decreases. Moreover $\lim_{n\to\infty} \dfrac{r\ln(n+r)}{\sqrt{n}} = 0$ for any $r > 1$. Therefore $f(n) < \ln 2$ when $n > n_r$ for a big enough n_r.

Example 8

$$\begin{aligned} r &= 2,\ n_r = 13 : C^2_{13+2} = 105 > \pi(13) = 101,\ \textit{but}\ \ C^2_{14+2} = 120 < \pi(14) = 135. \\ r &= 3,\ n_r = 29 : C^3_{29+3} = 4960 > \pi(29) = 4565,\ \textit{but}\ \ C^3_{30+3} = 5456 < \pi(30) = 5604. \end{aligned}$$

Appendix 3. Properties of vector-elementary functions [1]

1	Polynomial functions	Elementary in all variables everywhere being trivial examples of multivariate elementary functions (see item 7 for the non-trivial)
2	Rational functions	Elementary in all variables everywhere except its points of singularity, being trivial examples of multivariate elementary functions (see 7)
3	Conventional elementary functions and some special	Elementary
4	Composition of elementary in all variables vector-functions	Elementary in all variables (Theorem 2)
5	Inverse to an elementary in all variables vector-function	Elementary in all variables (Theorem 3)
6	An inverse function $X_n(x_1, ..., x_{n-1})$ defined in implicit equation $F(x_1, ..., x_n) = 0$, F is elementary in all variables	Elementary in all variables (Corollary 3)
7	Multivariate algebraic functions	Elementary at regular points in all variables exemplifying multivariate elementariness generally not expressible in rational functions
8	Derivatives of elementary function	Elementary
9	Integral $\int f(t, x)dt$, f elementary in t, x	Elementary in t, not necessarily in x. Only one such example (15) proven non-elementary.
10	Vector-function $u_k(t, x)$, a solution of an IVP $\{u'_k = f_k(u_1, ..., u_n, x)$, $\{f_k\}$ elementary in all variables	Elementary in t (Theorem 1), not necessarily in x. Analytical continuation in t is doable into the domain of regularity of the ODEs
11	$\lim\limits_{n\to\infty} f_n(t)$, all $f_n(t)$ elementary	Not necessarily elementary
12	$\sum_{n=0}^{\infty} a_n t^n$ converging to $f(t)$, a_n are arbitrary	Not necessarily elementary. Generally a method of analytical continuation not known.
13	$\sum_{n=0}^{\infty} a_n t^n$ converging to $f(t)$, a_n obtained via AD formulas for elementary ODEs	Elementary. Analytical continuation is doable via integration into the domain of regularity of the ODEs
14	Euler's Gamma function $\Gamma(x)$	Not elementary everywhere
15	$G(t,x) = \int_0^t u^{x-1}e^{-u}du$ generating $\Gamma(x)$ so that $\Gamma(x) = G(\infty, x)$	Elementary in t, non-elementary in x for all $t > 0$

Appendix 4. Unremovable and artificial singularities in ODEs

We have seen that regular functions (say $x(t) = t^n$ for natural n) may be solutions of both singular ODEs $(x' = \dfrac{nx}{t})$ and regular ones $(x' = nt^{n-1})$ so that the singularities in the respective ODEs were not a peculiarity of the proper functions, but something lateral.

We have seen also that some holomorphic entire functions (such as $x(t) = \dfrac{e^t - 1}{t}$, $x(0) = 1$) at certain points may be solutions of only singular ODEs: the singularities in those ODEs are unremovable, proper to those functions.

Here in this Appendix we are to demonstrate a bizarre fact that not only can singularities of ODEs be something lateral, but it is possible to artificially plant singularities at any regular points of ODEs. Here is how it is possible to purposefully corrupt regular ODEs.

Example 9 *Consider the simplest IVP for ODE* $x' = x,\quad x(0) = 1$, *for which it is known that* $x = x' = x'' = ... = x^{(n)} = ... = e^t$. *Observe, that for any point* a *this* $x(t)$ *satisfies also rational ODEs*

$$\begin{aligned} x'' &= x\frac{x - e^a}{x' - e^a}, \qquad x(0) = x'(0) = 1, \quad \textit{or} \\ x'' &= x\frac{x' - e^a}{x - e^a}, \end{aligned}$$

because for these initial values above $\dfrac{x - e^a}{x' - e^a} = \dfrac{e^t - e^a}{e^t - e^a} \equiv 1$. *However now the former holomorphic (linear) ODE turned into singular at an arbitrary point* a *while having the same holomorphic solution* e^t. *Note that for arbitrary variables* $x,\ x'$ *the fraction algebraically is irreducible. Only for the initial values given above and with the prior knowledge that* $x \equiv x'$ *the modified equations reduce to the original linear. For the initial conditions when* $x'(0) \neq 1$, *so that* $x \neq x'$ *the denominator may turn zero while the numerator is nonzero, so that both the ODE and its solution are singular at such point* a.

Theorem 2 *Let an IVP for a rational ODE*

$$x^{(n)} = \frac{P(t,\ x,\ ...,\ x^{(n-1)})}{Q(t,\ x,\ ...,\ x^{(n-1)})}, \quad x^{(k-1)}|_{t=0} = a_k \tag{25}$$

be regular at $t = 0$ so that its solution $x(t)$ is holomorphic at $t = 0$. Let $t = b \neq 0$ be some other point where $x(t)$ is holomorphic. Then there exist other ODEs singular at b yet having the same holomorphic solution $x(t)$.

Proof 3 *Differentiate (25) and obtain*

$$x^{(n+1)} = \frac{P_1(t,\ x,\ ...,\ x^{(n)})}{Q_1(t,\ x,\ ...,\ x^{(n)})}.$$

Then

$$\begin{aligned} x^{(n+1)} &= \frac{P_1}{Q_1} \cdot \frac{x^{(n)} - x^{(n)}|_{t=b}}{x^{(n)} - x^{(n)}|_{t=b}} = \\ & \frac{P_1}{Q_1} \cdot \frac{\frac{P}{Q} - x^{(n)}|_{t=b}}{x^{(n)} - x^{(n)}|_{t=b}}. \end{aligned}$$

Finally

$$x^{(n+1)} = \frac{P_1(t,\ x,\ ...,\ x^{(n)})(P(t,\ x,\ ...,\ x^{(n-1)}) - Q(t,\ x,\ ...,\ x^{(n-1)})x^{(n)}|_{t=b})}{Q_1(t,\ x,\ ...,\ x^{(n)})Q(t,\ x,\ ...,\ x^{(n-1)})(x^{(n)} - x^{(n)}|_{t=b})}.$$

This ODE is singular at an arbitrary chosen point b, yet its solution $x(t)$ is holomorphic at $t = b$.

Conclusion 4 *When we are looking for and find the target ODEs using an* arbitrary *method, the target ODE may happen to be singular due to various reasons such as having artificial singularity demonstrated above.*

Appendix 5. The special case which is proven

As Table 2 of the fundamental transforms shows, any polynomial system (7) may be transformed into the form with squares only. Therefore without losing generality the Conjecture may be reformulated for a square only system with m ODEs, but the proof is available only for $m = 2$.

Observe, that if the system in squares only is written as

$$\begin{aligned} x' &= F_1(x,\ y,\ z) = a_1x^2 + b_1y^2 + c_1z^2 \\ y' &= G_1(x,\ y,\ z) = a_2x^2 + b_2y^2 + c_2z^2 \\ z' &= H_1(x,\ y,\ z) = a_3x^2 + b_3y^2 + c_3z^2 \end{aligned}$$

then the Fundamental Sequence for it has simpler recurrent formulas

$$
\begin{aligned}
x' &= F_1 = a_1x^2 + b_1y^2 + c_1z^2 \\
& \quad \cdots \\
x^{(n+1)} &= F_{n+1} = \sum_{k=0}^{n} C_n^k (a_1F_kF_{n-k} + b_1G_kG_{n-k} + c_1H_kH_{n-k}) \\
& \quad \cdots
\end{aligned}
$$

where

$$
\begin{aligned}
F_0 &= x, \quad G_0 = y, \quad H_0 = z, \\
y^{(n+1)} &= G_{n+1} = \sum_{k=0}^{n} C_n^k (a_2F_kF_{n-k} + b_2G_kG_{n-k} + c_2H_kH_{n-k}) \\
z^{(n+1)} &= H_{n+1} = \sum_{k=0}^{n} C_n^k (a_3F_kF_{n-k} + b_3G_kG_{n-k} + c_3H_kH_{n-k}).
\end{aligned}
$$

Conjecture 1 *For every component (say u_1) of the IVP for polynomial system in squares only*

$$
\left\{ u_k' = \sum_{i=1}^{m} a_{ki}u_i^2 \ , \quad u_k|_{t=t_0} = b_k, \quad k = 1, ..., \ m \right.
$$

there exists an IVP for $n-$order rational ODE

$$
u^{(n)} = \frac{P(t, \ u, \ u', ..., \ u^{(n-1)})}{Q(t, \ u, \ u', ..., \ u^{(n-1)})}, \quad Q(t, \ u, \ u', ..., \ u^{(n-1)})\big|_{t=t_0} \neq 0
$$

regular at $t = t_0$ and having u_1 as a unique solution.

Proof 4 *(For $m = 2$ only). Consider a system*

$$
\begin{aligned}
x' &= a_1x^2 + b_1y^2 \\
y' &= a_2x^2 + b_2y^2.
\end{aligned}
$$

If $b_1 = 0$, the target ODE is $x' = a_1x^2$ which proves the Conjecture. Otherwise assume $b_1 \neq 0$.

$$
\begin{aligned}
x' &= a_1x^2 + b_1y^2, \qquad y^2 = (x' - a_1x^2)/b_1 \\
y' &= a_2x^2 + b_2y^2, \qquad y' = a_2x^2 + \frac{b_2}{b_1}(x' - a_1x^2)
\end{aligned}
$$

Looking at y', observe that all derivatives $y^{(n)}$ depend on x and its derivatives only, and they may be expressed as polynomials G_n: $y^{(n)} = G_n(x,\ x', ...\ x^{(n)})$, $n = 1,\ 2, ...$ Utilizing this, differentiate the first equation:

$$\begin{aligned} x^{(n+1)} &= a_1(x^2)^{(n)} + b_1(y^2)^{(n)} = \\ &= a_1(x^2)^{(n)} + 2b_1(yy^{(n)} + ny'y^{(n-1)} + ...) \\ &= a_1(x^2)^{(n)} + 2b_1(yG_n + nG_1G_{n-1} + ...), \qquad n = 1,\ 2, ... \end{aligned}$$

Now observe that y occurs only in one monomial: the one with factor $G_n(x,\ x', ...\ x^{(n)}) = y^{(n)}$. If at least one $y^{(n)}|_{t=t_0} \neq 0,\ n = 1,\ 2, ...,$ then y may be eliminated and substituted in the following equations for bigger n, so that we can obtain infinitely many rational ODE's in x regular at $t = t_0$. Otherwise, if all $G_n|_{t=t_0} = y^{(n)}|_{t=t_0} = 0,\ \ n = 1,\ 2, ...,$ then y must be a constant so that the first ODE takes form $x' = a_1x^2 + const$. That concludes the proof.

Unfortunately, for $m > 2$ this method of proof does not work. However this method applies for a more general system.

Proof 5 *Consider a system*

$$\begin{aligned} x' &= p_1(t,\ x) + b_1y^2 \\ y' &= p_2(t,\ x) + q_2(t,\ x)y^2 \end{aligned}$$

where $p_1,\ p_2,\ q_2$ are arbitrary polynomials in $t,\ x$ and $b_1 \neq 0$. Then

$$\begin{aligned} x' &= p_1(t,\ x) + b_1y^2 \qquad & y^2 = (x' - p_1(t,\ x))/b_1 \\ y' &= p_2(t,\ x) + q_2(t,\ x)y^2 \qquad & y' = p_2(t,\ x) + \frac{q_2(t,\ x)}{b_1}(x' - p_1(t,\ x)). \end{aligned}$$

Again observe that all derivatives $y^{(n)}$ depend on t, x and its derivatives only, and they may be expressed as polynomials G_n: $y^{(n)} = G_n(t,\ x,\ x', ...\ x^{(n)})$, $n = 1,\ 2, ...$ Utilizing this, differentiate the first equation:

$$\begin{aligned} x^{(n+1)} &= (p_1(t,\ x))^{(n)} + b_1(y^2)^{(n)} = \\ &= (p_1(t,\ x))^{(n)} + 2b_1(yy^{(n)} + ny'y^{(n-1)} + ...) = \\ &= (p_1(t,\ x))^{(n)} + 2b_1(yG_n + nG_1G_{n-1} + ...), \qquad n = 1,\ 2, ... \end{aligned}$$

The rest of the proof is the same as before.

References

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2. Gofen, A., (2008), Unremovable 'Removable' Singularities, *Complex Variables and Elliptic Equations*, Vol. 53, No. 7, p. 633-642. Also here.

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4. Gofen, A. (2013), Powers which commute or associate as solutions of ODEs. *Teaching Mathematics and Computer Science*, 11/2, p. 241-254. Also here.

5. Unsolved problems.