The Taylor Center ODEs Solver as a Generator of Simulations and Phase Portraits

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All-in-One GUI software running under Windows with the high accuracy of integration up to 63 binary digits and sophisticated graphics with unique features such as …

- Real-time animation of trajectories for 2D (and 3D stereo vision using the Red/Blue glasses),
- Phase portrait designer,
- Generator of sequences of frames for movies,
- Recording scripts of real-time simulation for lectures and presentations.

(Here is the full list of features)

Abstract

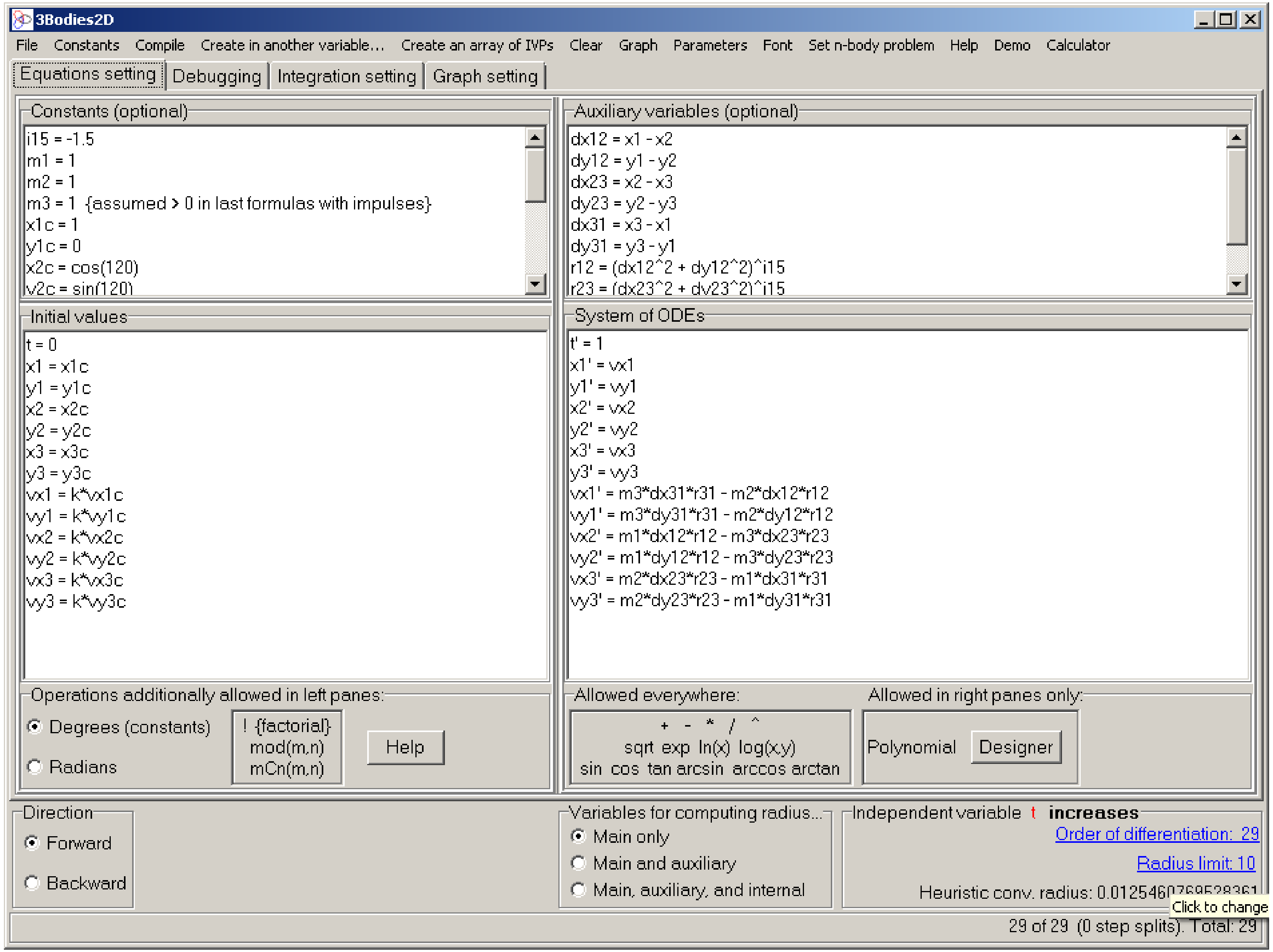
First, the only goal of this All-in-One Taylor integrator [1] for challenging ODEs was scientific research and numeric experimentation [2, 3].

Later, I began collecting illustrious problems in applied ODEs – now a vast list [4] of simulations (without textbooks indeed).

Yet the very method of the real-time graphics suggested to utilize it as a generator of simulations or a virtual lab (Exploratorium [5]). Now it consists of only a few topics in my own expertise: didactic texts with directives what to watch in the respective simulations. When readers encounter a name of a simulation, there is a mechanism for running it by selecting its name.

In fact, any textbook for applied ODEs may be enriched with such simulations. Consequently, instead of still images, the textbook will provide a live real-time animation. And this is the main message of my talk today, and of the Workshop [6].

The front panel of the Taylor solver



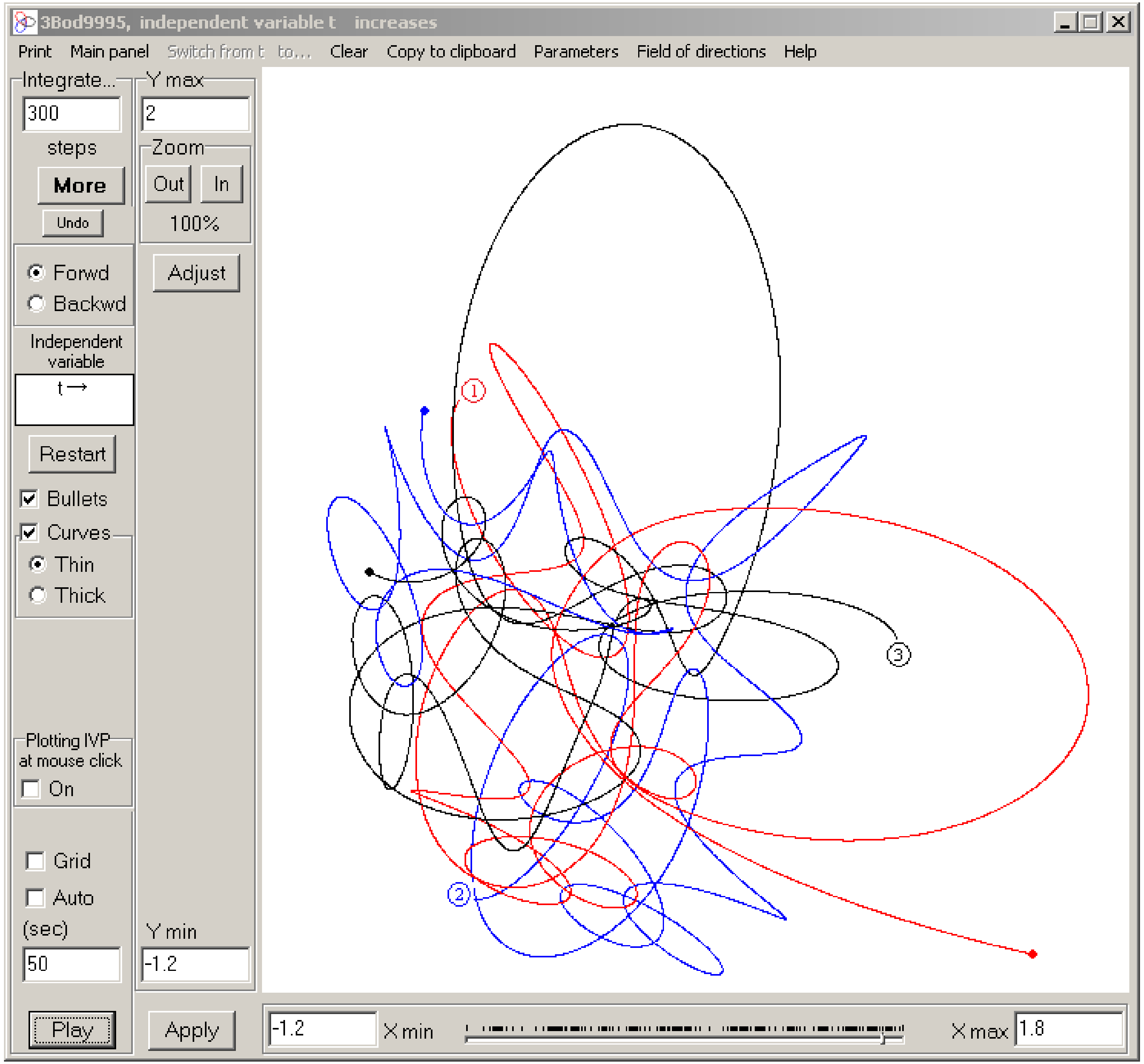
Differential equations for the 3-body problem

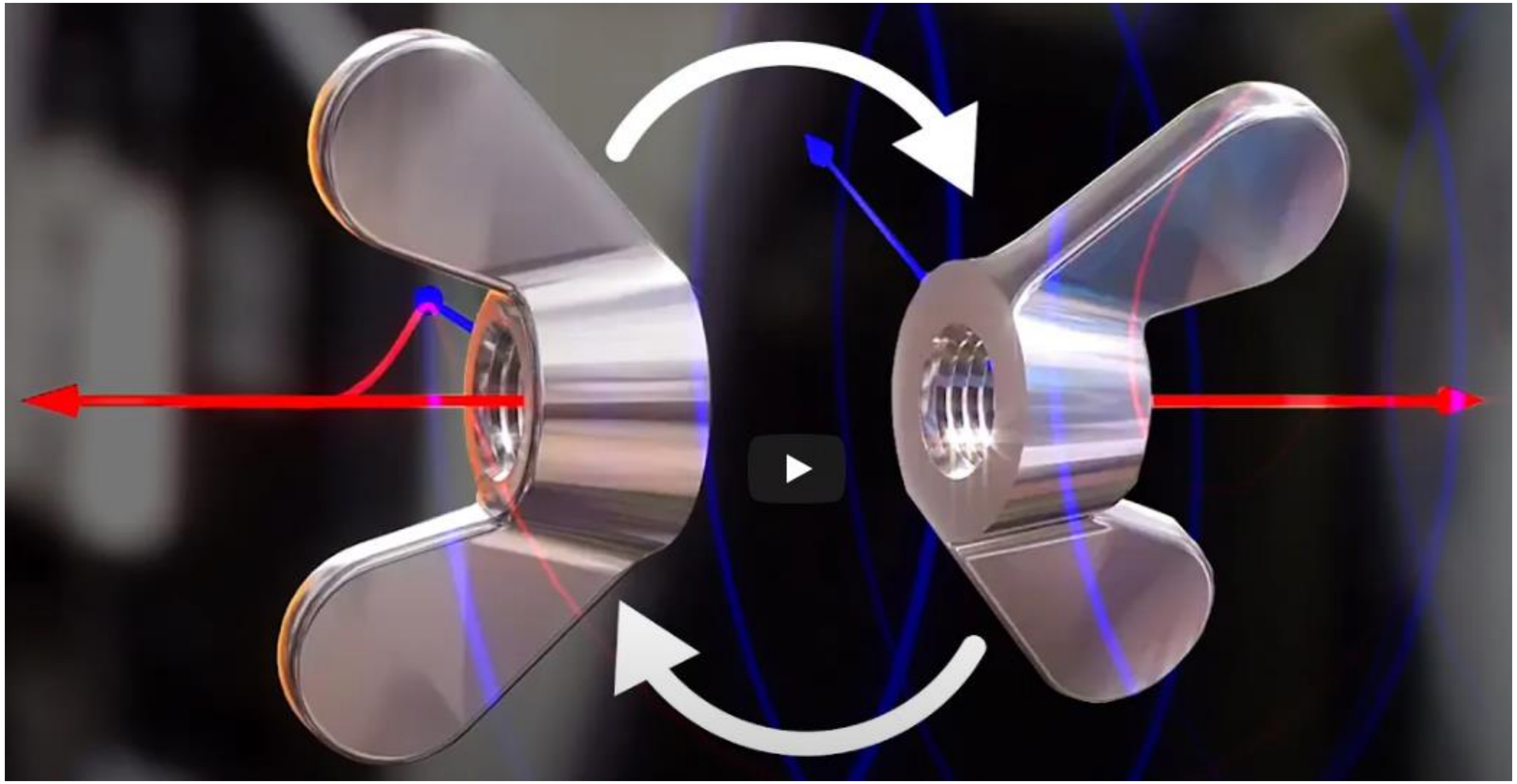
This is a still image of trajectories of the disturbed Lagrange case of the 3-body problem.

However, the software can display a dynamic real-time animation of the motion along these trajectories demonstrating acceleration and deceleration of the bodies.

The very method of such graphing suggested an idea to utilize it as a generator of simulations or a virtual lab – [Exploratorium] – for studying and teaching applied mathematics. The Phase Portraits present great scientific and teaching value in itself. That is why the employment of this software as a generator of simulations for teaching became a separate [goal of the website] and of this talk.

In perspective, it's a good idea to animate trajectories considered in all textbooks for applied ODEs.





The Dzhanibekov effect – a special case of the free rigid body motion spinning around the *middle* axis of momentum of inertia. An instability of spin around the middle axis has been known since the 19th century.

However, the remarkable appearance of this instability was discovered only much later by astronaut Dzhanibekov, who first observed this effect under weightlessness in the Soviet space station.

During routine maintenance, Vladimir Dzhanibekov was unscrewing a wingnut sitting on a bolt fixed to the wall of the station. When the nut was near the end of the bolt, he set it into a fast spin so that it separated from the bolt keeping on its fast spinning in air. However, after several seconds, all of a sudden, the wingnut changed its orientation to the opposite while continuing its spin. And then, in a few seconds, it flipped again. And so it went on and on (click on the picture to view a video lecture about the Dzhanibekov story).

The Exploratorium offers a mathematical explanation of this phenomenon.

Below is the system of ODEs modeling a free rigid body in a laboratory system of coordinates in the three Euler variables (courtesy to Dimitry Garanin). The remarkable property of this system (which made this research possible) is that the ODEs for θ and ψ comprise a stand-alone system admitting a plane phase portrait.

The ODEs (86) in [2] for a free asymmetric top in original form are

$$\begin{aligned}
\dot{\theta} &= \left(\frac{1}{I_1} - \frac{1}{I_2}\right) L \sin\theta \sin\psi \cos\psi \\
\dot{\phi} &= \left(\frac{\sin^2\psi}{I_1} + \frac{\cos^2\psi}{I_2}\right) L \\
\dot{\psi} &= \left(\frac{1}{I_3} - \frac{\sin^2\psi}{I_1} - \frac{\cos^2\psi}{I_2}\right) L \cos\theta = \left(\frac{L}{I_3} - \dot{\phi}\right) \cos\theta
\end{aligned}$$

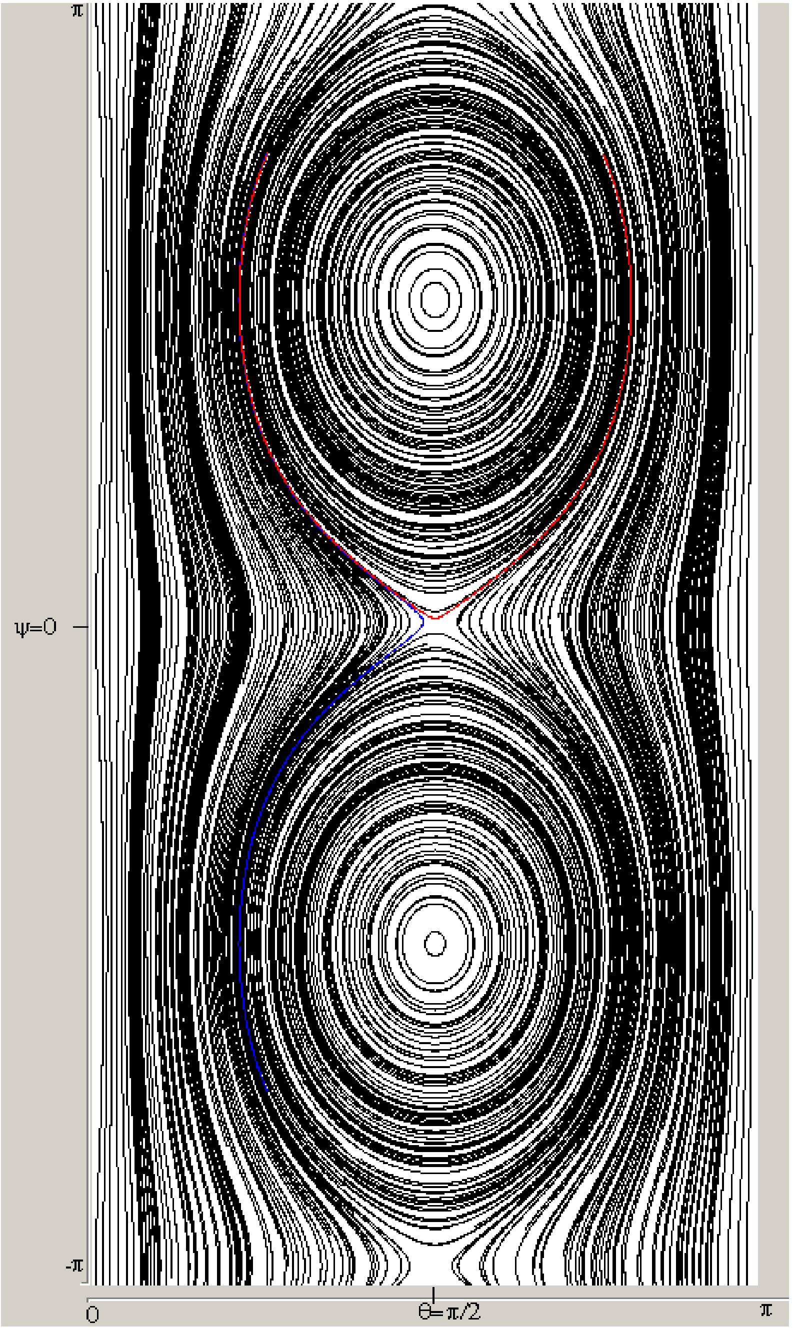
Observe that the ODEs for $\dot{\theta}$ and $\dot{\psi}$ alone comprise a consistent system of two ODEs, which makes it possible later to construct a plane phase portrait $\{\theta(t),\ \psi(t)\}$ for this sub-system (Fig. 1).

Here is a phase portrait for the variables θ and ψ showing that points (θ=π/2, ψ=kπ) are unstable rest points. If the θ and ψ of the body were exactly at one of these rest points, the body would stay at this point infinitely long spinning without flipping.

However, due to a small disturbance, the θ and ψ of the body are always in a small vicinity of the points (θ=π/2, ψ=kπ) – and there are 4 trajectories and 8 ways leading to one of the neighboring rest points: two internal trajectories (one of which is shown in red), and two external trajectories (one of which is shown in blue). Whichever way the body randomly chooses, it gets to one of the neighboring points and flips.

In order to visualize the real-time dynamic of the process, there are simulations in the Exploratorium, which show the motion of a dot along one of the trajectories. There a viewer can see, that first the dot stays still at one of the points (θ=π/2, ψ=kπ). Then the point jumps in a fast swing to the next point along one of the 4 trajectories and rests there again for some time.

The periods of rest correspond to simple spinning of the body, while the jumps mean that the body flips according to the Dzhanibekov effect. Therefore, the still image of the phase portrait plus the real-time simulation along the trajectories provide a full explanation of the Dzhanibekov effect.



References

1. The software. http://TaylorCenter.org/Gofen/TaylorMethod.htm
2. R. Montgomery. Dropping bodies, in "The Mathematical Intelligencer", 2023. https://link.springer.com/content/pdf/10.1007/s00283-022-10252-4.pdf (The Taylor Solver used for research)
3. M. Frenkel et al. The Continuous Measure of Symmetry as a Dynamic Variable… Symmetry 2023, 15, 2153. https://doi.org/10.3390/sym15122153 (The Taylor Solver used for research)
4. Simulations (though without textbooks), http://TaylorCenter.org/Gofen/Teaching/Samples.htm
5. Exploratorium, i.e. textbooks with the respective simulations, http://TaylorCenter.org/Exploratorium
6. The Workshop, http://TaylorCenter.org/Workshops.htm

Thank you for the attention