Outlining Linked Tori

(Menu item *Demo/Linked tori,* or script *Tori.scr,* and *ToriArray.ode*)

Problem:

Outlining linked tori. Points $(x_1,\ y_1,\ z_1)$ on a surface of a torus may be defined by the parametric equations

$$\begin{aligned} x_1 &= (R + r\cos\varphi)\cos\omega \\ y_1 &= (R + r\cos\varphi)\sin\omega \\ z_1 &= r\sin\varphi \end{aligned} \tag{1}$$

and then the torus linked with the first one ("perpendicular" to it) is defined by equations

$$\begin{aligned} x_2 &= x_1 + R \\ y_2 &= z_1 \\ z_2 &= y_1 \end{aligned}$$

We can outline these two surfaces using various families of curves.

First method: a screw line on a torus.

Here is the equations of a screw line for the first torus depending on the parameter t:

$$\begin{aligned} x_1 &= (R + r\cos\varphi t)\cos\omega t \\ y_1 &= (R + r\cos\varphi t)\sin\omega t \\ z_1 &= r\sin\varphi t. \end{aligned}$$

Go to *Demo/Linked tori,* or open script *Tori.scr.* Experiment with different angles viewing these linked tori.

Second method: families of circles on a torus.

As it follows from the equation (1), we can generate two families of circles on a torus perpendicular to each other

$$\begin{aligned} x_{1k} &= (R + r\cos t)\cos\omega_k \\ y_{1k} &= (R + r\cos t)\sin\omega_k \\ z_{1k} &= r\sin t \end{aligned}$$

and

$$\begin{aligned} x_{2l} &= (R + r\cos\varphi_l)\cos t \\ y_{2l} &= (R + r\cos\varphi_l)\sin t \\ z_{2l} &= r\sin\varphi_l \end{aligned}$$

for sets of constants ω_k and φ_l. Correspondingly we define the set of auxiliary variables

$$\left.\begin{array}{l} \text{u =cos(t)} \\ \text{v = sin(t)} \\ \text{x02 = (RBig + r*u)*cos(w)} \\ \text{y02 = (RBig + r*u)*sin(w)} \\ \text{z02 = r*v} \\ \text{x20 = x02 + Rbig} \\ \text{y20 = z02} \\ \text{z20 = y02} \\ \text{x01 = (RBig + r*cos(fi))*u} \\ \text{y01 = (RBig + r*cos(fi))*v} \\ \text{z01 = r*sin(fi)} \\ \text{x10 = x01 + Rbig} \\ \text{y10 = z01} \\ \text{z10 = y01} \end{array}\right\} \text{One set of 4 curves} \qquad (2)$$

specifying for every of the parameters w (standing for ω_k) and fi (for φ_l) the 4 curves: 2 big circles $\{x_{02},\ y_{02},\ z_{02}\} \perp \{x_{20},\ y_{20},\ z_{20}\}$ and 2 small circles $\{x_{01},\ y_{01},\ z_{01}\} \perp \{x_{10},\ y_{10},\ z_{10}\}$.

Now in order to obtain the two families of the circles on each of the tori we specify a multi-valued IVP

$$\begin{array}{ll} t = 0 & t' = 1 \\ fi = 0;\ \text{Pi2};\ 10\ \text{n} & fi' = 0 \\ w = 0;\ \text{Pi2};\ 20\ \text{n} & w' = 0 \end{array}.$$

Here all three ODEs are trivial: the one for t' specifies the independent variable of integration forcing each of the curves to evolve. The ODEs for fi' and w' define dummy "variables" that are constants. For them the initial values are specified as multi-values - the nodes of regular grids: 10 nodes between 0 and 2π for fi, and 20 nodes between 0 and 2π for w.

What to do.

First let's plot just one set of 4 curves (2).

File/Open *ToriArray.ode* (opening ODEs, not scripts!) which displays the above ODEs. This is a multi-valued IVP therefore it cannot be compiled as is. First use the commenting {} to nullify inappropriate characters in order to turn the multi-valued IVP into a single-valued

fi =0{; Pi2; 10n}
w =0{; Pi2; 20n}

and compile it. This will bring you to the Graph setting page for selection of the curves to graph. We are going to have a 3D picture, therefore click the 3D radio button. Due to the name choice for our variable of interest according to the convention, all the curves we want to plot have the form $\{x_k,\ y_k,\ z_k\}$ whose names appear in the lower part of the table. For such special name selection, this program offers an automation allowing to automatically specify all 4 curves by clicking only one button $\boxed{\{x1, y1\}, ... >>}$ - and all four appear in the right pane. Click *Apply* button.

This action opens the Graph window containing the result of 10 (by default) steps of integration. This is a stereo image to be viewed via a pair of Red/Blue glasses. Put them on (over the optical glasses used by you), slightly move the α handler to turn the 3D image, and observe 4 circles: 2 reciprocally perpendicular big circles serving as latitudes of linked tori, and 2 reciprocally perpendicular small circles serving as meridians of linked tori. That is the image generated by only one IVP for only one set of 4 such lines which obviously are not enough to outline linked tori. In the next step we are going to clone this IVP for various sets of initial points such that two families of latitudes and meridians be plotted.

Cloning.

Go to the main window *Equation setting* page and remove commenting {} in the *Initial values.* making this IVP multi-valued again. It needs first to be unfolded into an aggregate system of cloned ODEs. Go to the menu item *Create an arrays of IVPs* and choose *Create.* This opens the window for specifying which kind of array to create. We have set 10 values $\varphi_0,\ \varphi_1, ...,\ \varphi_9$ and 20 values $\omega_0,\ \omega_1, ..., \omega_{19}$. Of those the program offers to build either a direct product, or a set of corresponding components of vectors. Choose a set of corresponding components (even though the numbers of values ω and φ are not equal). The set of corresponding components in this case will be generated like this:

$$\begin{array}{ll} \varphi_0 & \omega_0 \\ \varphi_1 & \omega_1 \\ ... & ... \\ \varphi_9 & \omega_9 \\ \varphi_9 & \omega_{10} \\ \varphi_9 & \omega_{11} \\ ... & ... \\ \varphi_9 & \omega_{19} \end{array}.$$

These are initial points for 20 curves (circles). All circles are going to be closed, therefore such a bizarre choice of the initial points will not affect the final appearance of the closed circle families on the tori.

Click *Unfold* button, which causes creation of 19 clones with the specified initial values comprising the aggregate single value IVP. Compile it, which brings you to the Graph setting page for selection the curves to graph. (This time the 3D radio button is already on). Again, due to the name choice for the variable of interest, all the curves we want to plot have the form $\{x_k,\ y_k,\ z_k\}$ whose names appear in the lower part of the table. Clicking the button $\boxed{\{x1, y1\}, ... >>}$ you will get all 4×20=80 curves. Click the *Apply* button.

Now you are in Graph window displaying the 3D stereo image. Like before, put on the stereo glasses and slightly move the α handler to turn the 3D image. Now you must see two tori each outlined by a family of latitudes and meridians, as specified in the multi-valued setting for this array of IVPs.

The Cornu spiral

(Menu item *Demo/Spirals/Cornu* or script *CornuSpiral.scr*)

The mathematical definition of the Cornu spiral is given in the natural parametric coordinates

$$k = s$$

where s is an independent parameter of the arch length, and k - the curvature. In the Cartesian coordinates (x, y)

$$\begin{aligned} x &= \int_0^t \cos\left(\frac{t^2}{2}\right) dt \\ y &= \int_0^t \sin\left(\frac{t^2}{2}\right) dt. \end{aligned}$$

Verify this representation. As $(x')^2 + (y')^2 = 1$,

$$\begin{aligned} k &= \frac{x'y'' - x''y'}{\left((x')^2 + (y')^2\right)^{3/2}} = x'y'' - x''y' = \\ &= \cos\left(\frac{t^2}{2}\right)\cos\left(\frac{t^2}{2}\right)t - \left(-\sin\left(\frac{t^2}{2}\right)t\right)\sin\left(\frac{t^2}{2}\right) = t, \\ s &= \int_0^t \sqrt{(x')^2 + (y')^2} dt = t \end{aligned}$$

so that really $k = s = t$.

Remark 1 *In this parametric example, the ODEs play more significant role (than to merely provide $t' = 1$), because both x and y require antiderivatives.*

We introduce auxiliary variables

$$\begin{aligned} \text{t2=t\^{}2/2} \\ \text{cos2=cos(t2)} \\ \text{sin2=sin(t2)} \end{aligned}$$

so that the ODEs are

$$\begin{aligned} t' &= 1 \\ x' &= \text{cos2} && \{\text{Integral of } \cos(\text{t}^2/2)\} \\ y' &= \text{sin2} && \{\text{Integral of } \cos(\text{t}^2/2)\} \\ x1' &= \text{-cos2} \\ y1' &= \text{-sin2} \end{aligned}$$

In this example non-trivial ODEs for x and y are needed because the auxiliary variables represent their antiderivatives.

The Curly spiral

(Menu item *Demo/Spirals/Curls* or script *CurlySpiral.scr*)

The parametric equations of the curly spiral is

$$
\begin{aligned}
x &= \int_0^t \cos\left(\sin(t) - t\cos t\right) dt \\
y &= -\int_0^t \sin\left(\sin(t) - t\cos t\right) dt.
\end{aligned}
$$

Correspondingly we introduce auxiliary variables

u = sin(t) - t*cos(t)

x1 = -x

Here varaible x1 is introduced for having the mirrored graph. Then the ODEs are

t' = 1

x' = cos(u)

y' = -sin(u)

In this example too the non-trivial ODEs are needed because x and y are represented as antiderivatives.