Theorems for the orbits of a photon near a black hole

Recalling that $u = \dfrac{1}{r}$ and $r > 2GM$ (outside the horizon) so that $u < \dfrac{1}{2GM}$ and $1 - 2GMu > 0$, the following takes place.

THEOREM 2: Given that the parameter b is defined only outside the horizon so that $1 - 2GMu > 0$, and its formula is

$$b = \frac{r\sin\alpha}{\sqrt{1 - \dfrac{2GM}{r}}} = \frac{\sin\alpha}{u\sqrt{1-2GMu}}, \quad \text{and} \quad \frac{1}{b^2} = \frac{u^2(1-2GMu)}{\sin^2\alpha},$$

the under-the-root expression $f(u) = \dfrac{1}{b^2} + 2GMu^3 - u^2$ in formula (5)

$$\frac{du}{d\varphi} = u' = \pm\sqrt{\frac{1}{b^2} + 2GMu^3 - u^2} \qquad ((5))$$

is always non-negative

PROOF: Consider

$$f(u) = \frac{1}{b^2} + 2GMu^3 - u^2 = \frac{1}{b^2} - u^2(1-2GMu)$$

and apply the formula for $\dfrac{1}{b^2}$:

$$f(u) = \frac{u^2(1-2GMu)}{\sin^2\alpha} - u^2(1-2GMu) = u^2(1-2GMu)\left(\frac{1}{\sin^2\alpha} - 1\right) \geq 0$$

- because all the factors are non-negative. The equality is reached when...

- $\sin^2\alpha = 1$, i.e. when the velocity vector of a photon is perpendicular to the radius $r(\varphi)$, which happens only when the photon is outside the circular orbit reaching the closest distance to the center. Or...
- $u = 0$ which happens at infinity, or...
- $u = \dfrac{1}{2GM}$, i.e. at the horizon.■

LEMMA: The function $g(u) = 2GMu^3 - u^2 \leq 0$ (while $u \in \left[0; \dfrac{1}{2GM}\right]$) reaches its minimum $-\dfrac{1}{27(GM)^2}$ at $u = \dfrac{1}{3GM} < \dfrac{1}{2GM}$, or $r = 3GM$ (i.e. on the circular orbit).

PROOF: The derivative $g'(u) = 6GMu^2 - 2u$, $g'(u) = 0$ for $u_{circ} = \dfrac{1}{3GM}$ so that $g(u_{circ}) = -\dfrac{1}{27(GM)^2} = -\dfrac{1}{b_{crit}^2}$. ■

Inbound motion: from infinity toward the black hole

Motion of a photon from infinity toward the black hole means that $u = \dfrac{1}{r}$ increases from zero to its maximum $u_{\max}$ which may be either inside the circular orbit (capture), or outside it (bypassing the black hole). In order to specify the inbound motion, we must set the initial value for u' in fromula (5) choosing the "+" sign.

Case 1 *Bypassing a black hole:* $\boxed{b > b_{crit} = 3GM\sqrt{3}.}$ *As u grows from* 0 *while* $g(0) = 0$ *and decreases,* $f(u) = \dfrac{1}{b^2} + g(u)$ *decreases from* $\dfrac{1}{b^2}$ *until it reaches* $f_{\min} = f(u_{circ}) \geq 0$ *by the Lemma. We are to prove that* $f_{\min} = 0$ *reaching zero at* $u_{\max} < u_{circ}$ *(*u_{circ} *delivers minimum to* $g(u)$ *in the Lemma).*

$$g(u) > -\frac{1}{b_{crit}^2},$$

then

$$f(u) = \frac{1}{b^2} + g(u) > \frac{1}{b^2} - \frac{1}{b_{crit}^2}.$$

However,

$$\frac{1}{b^2} - \frac{1}{b_{crit}^2} < 0$$

strictly at $u = u_{circ}$*. As* $f(u)$ *cannot be negative by Theorem 2, after reaching* 0 *at some* $u_{\max} < u_{circ}$*,* $f(u)$ *must increase. Therefore,* $u_{\max}$ *is the closest approach of the photon,* $u_{\max} < u_{circ}$*. After reaching* $u_{\max}$*, u diminishes because the sign at the root must be changed from "+" to "-" in the ODE (5)* $u' = \pm\sqrt{f(u)}$*; u diminishes from* $u_{\max}$ *to* 0 *while* $f(u)$ *increases along the same graph backword.*

Case 2 *Capture by a black hole:* $\boxed{b < b_{crit} = 3GM\sqrt{3}.}$ *Following the steps of Case 1, now, however, we come to the opposite inequality*

$$\frac{1}{b^2} - \frac{1}{b_{crit}^2} > 0$$

meaning that $f(u)$ *never reaches zero. With the sign "+" in (5), u keeps monotonously increasing so that the photon moving along a spiral crosses the circular orbit and then also the horizon.*

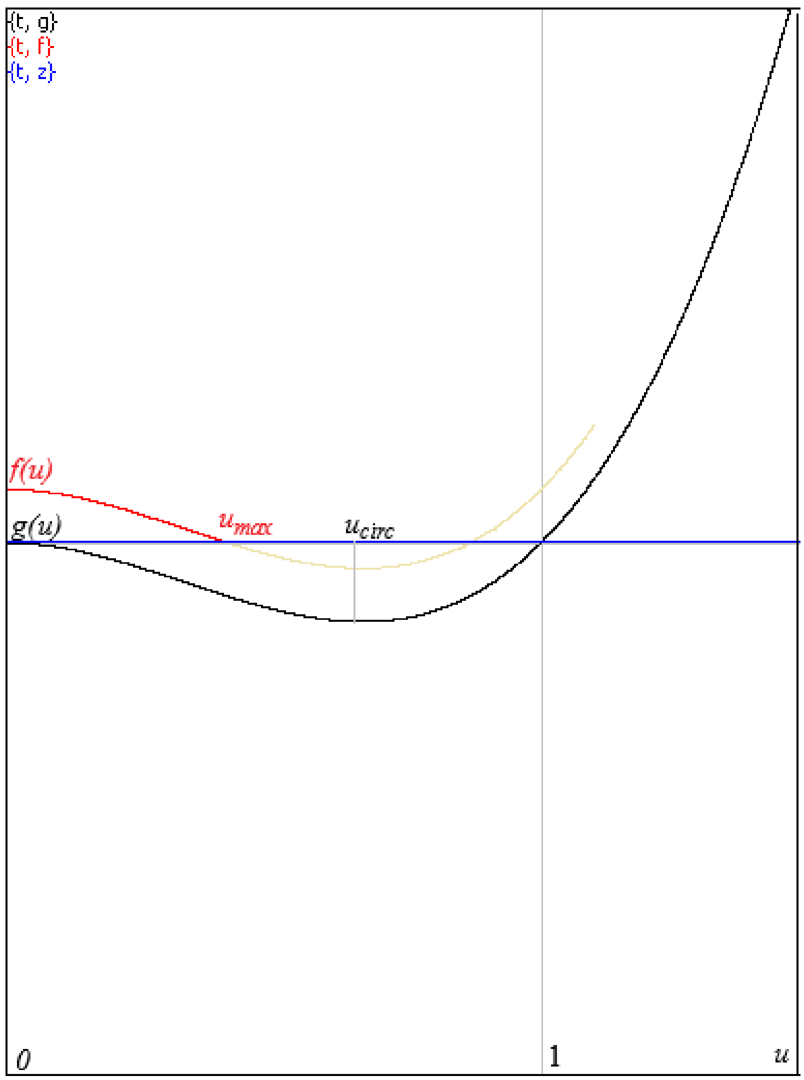

Figure 1: Case 1. Bypassing a black hole: $b > b_{crit}$

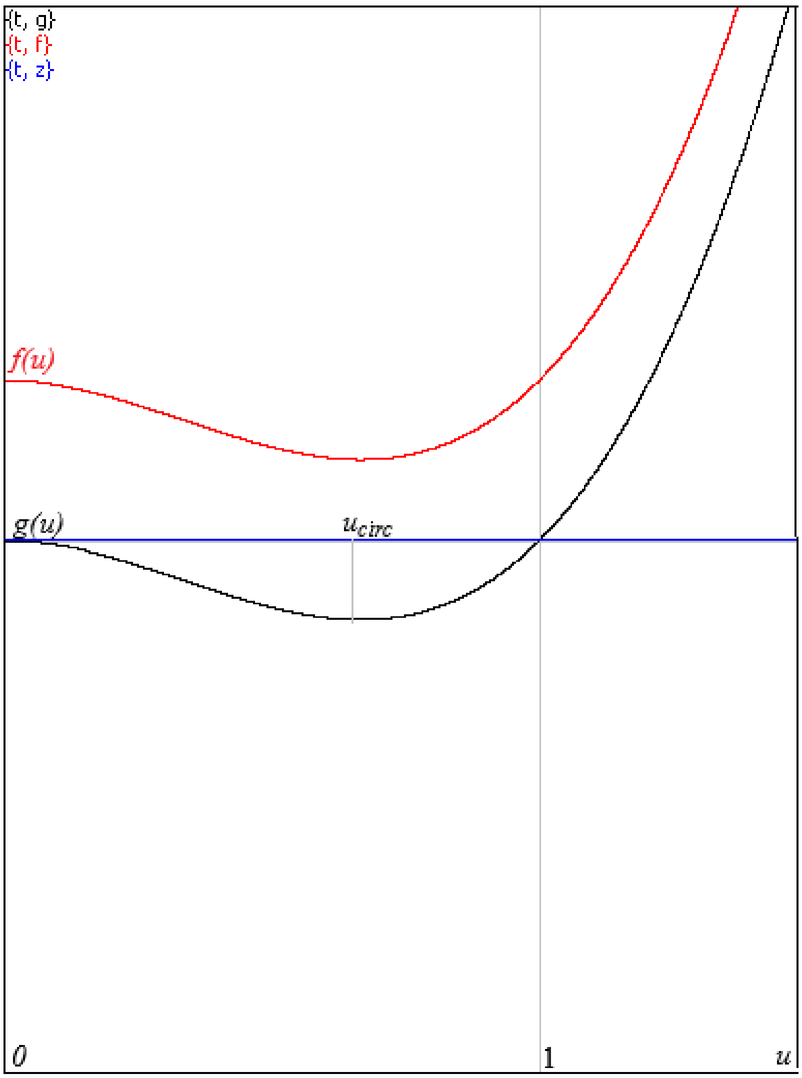

Figure 2: Case 2. Capture by a black hole: $b < b_{crit}$

Case 3 *Capture by the circular orbit:* $\boxed{b = b_{crit} = 3GM\sqrt{3}.}$ *Unlike the Cases 1, 2, now*

$$\frac{1}{b^2} - \frac{1}{b_{crit}^2} = 0.$$

Again, $f(u) = \dfrac{1}{b^2} + g(u)$ *keeps decreasing from* $\dfrac{1}{b^2}$ *towards* 0*, but it never reaches it, as it follows from the next ...*

THEOREM 3: If $b = b_{crit}$ and a photon moves from infinity toward a black hole, u increases and $f(u)$ decreases but never reaches 0, meaning that the photon approaches the circular orbit along an inifinit spiral but never crosses or tauches the sircular orbit.

PROOF: Suppose the opposite, i.e. that $f(u)$ does reach zero at some point $(\varphi,\ r_{circ})$ on the circumference of the circular orbit. Then , at this point φ, we have that $u' = 0$ by formula (5), and $u = \dfrac{1}{r_{circ}}$. However, according to Theorem 1, there is another solution for these very initial values: the exact circular solution. Both these solutions satisfy a regular ODE (6)

$$u'' = 3GMu^2 - u \tag{(6)}$$

which cannot have two solutions for the same initial values. This controvesy proves the Theorem. ■

CONCLUSION: The case of $b = b_{crit}$ is remarkable in that the incoming photon is caught by the circular orbit rather than by the horizon, winding around the circumference in a tight spiral. As we will see further, outgoing photons moving from a vicinity near the horizon away having the parameter $b = b_{crit}$, are caught into an infinite spiral approaching the circular orbit from inside too.

Outbound motion: from near the horizon to infinity

An attempt of motion of a photon from near the horizon away (possibly to infinity) means that $u = \dfrac{1}{r}$ initially decreses from initial value to its minimum $u_{\min}$ which may be either inside the circular orbit (capture), or outside it (escape). In order to specify the outbound motion, we must set the initial value for u' in fromula (5) choosing the "-" sign.

As we will see from the solutions of ODE (5), it's possible for a photon to escape out even from near proximity of the horizon depending on the direction into which it was fired. For example, if photon was fired along the radius (i.e. $b = 0$), it escapes into infinity.