```
> restart;
```

Generate the ODEs for the 3D spherical double pendulum.

```
> with(linalg) :
>
> x[1] := L[1]·sin(theta[1](t))·cos(phi[1](t));
```

$$x_1 := L_1 \sin(\theta_1(t)) \cos(\phi_1(t)) \quad (1)$$

```
> y[1] := L[1]·sin(theta[1](t))·sin(phi[1](t));
```

$$y_1 := L_1 \sin(\theta_1(t)) \sin(\phi_1(t)) \quad (2)$$

```
> z[1] := −L[1]·cos(theta[1](t));
```

$$z_1 := -L_1 \cos(\theta_1(t)) \quad (3)$$

```
> simplify(x[1]^2 + y[1]^2 + z[1]^2);
```

$$L_1^2 \quad (4)$$

```
>
> x[2] := x[1] + L[2]·sin(theta[2](t))·cos(phi[2](t));
```

$$x_2 := L_1 \sin(\theta_1(t)) \cos(\phi_1(t)) + L_2 \sin(\theta_2(t)) \cos(\phi_2(t)) \quad (5)$$

```
> y[2] := y[1] + L[2]·sin(theta[2](t))·sin(phi[2](t));
```

$$y_2 := L_1 \sin(\theta_1(t)) \sin(\phi_1(t)) + L_2 \sin(\theta_2(t)) \sin(\phi_2(t)) \quad (6)$$

```
> z[2] := z[1] − L[2]·cos(theta[2](t));
```

$$z_2 := -L_1 \cos(\theta_1(t)) - L_2 \cos(\theta_2(t)) \quad (7)$$

```
> simplify(x[2]^2 + y[2]^2 + z[2]^2);
```

$$2 \sin(\theta_2(t)) L_1 L_2 \left(\sin(\phi_1(t)) \sin(\phi_2(t)) + \cos(\phi_2(t)) \cos(\phi_1(t))\right) \sin(\theta_1(t)) + 2 \cos(\theta_1(t)) \cos(\theta_2(t)) L_1 L_2 + L_1^2 + L_2^2 \quad (8)$$

```
> PE := g·(m[1]·z[1] + m[2]·z[2]);
```

$$PE := g \left(-m_1 L_1 \cos(\theta_1(t)) + m_2 \left(-L_1 \cos(\theta_1(t)) - L_2 \cos(\theta_2(t))\right)\right) \quad (9)$$

```
> KE[1] := m[1]/2 · simplify(diff(x[1], t)^2 + diff(y[1], t)^2 + diff(z[1], t)^2);
```

$$KE_1 := \frac{m_1 L_1^2 \left(-\left(\frac{d}{dt} \phi_1(t)\right)^2 \cos(\theta_1(t))^2 + \left(\frac{d}{dt} \theta_1(t)\right)^2 + \left(\frac{d}{dt} \phi_1(t)\right)^2\right)}{2} \quad (10)$$

```
> KE[2] := m[2]/2 · simplify(diff(x[2], t)^2 + diff(y[2], t)^2 + diff(z[2], t)^2);
```

$$KE_2 := \frac{1}{2} \Bigg(m_2 \Bigg(\left(\frac{d}{dt} \theta_1(t)\right)^2 L_1^2 + 2 L_1 \Bigg(\Big(\cos(\theta_2(t)) \big(\sin(\phi_1(t)) \sin(\phi_2(t)) \quad (11)$$

$$+\cos(\phi_2(t))\cos(\phi_1(t)))\cos(\theta_1(t))+\sin(\theta_1(t))\sin(\theta_2(t)))\left(\frac{d}{dt}\theta_2(t)\right)$$

$$+\cos(\theta_1(t))\sin(\theta_2(t))\left(\frac{d}{dt}\phi_2(t)\right)(\sin(\phi_1(t))\cos(\phi_2(t))-\sin(\phi_2(t))\cos(\phi_1(t))))$$

$$L_2\left(\frac{d}{dt}\theta_1(t)\right)+\left(\frac{d}{dt}\theta_2(t)\right)^2L_2^2-2\cos(\theta_2(t))\sin(\theta_1(t))\left(\frac{d}{dt}\right.$$

$$\left.\phi_1(t)\right)L_1L_2(\sin(\phi_1(t))\cos(\phi_2(t))-\sin(\phi_2(t))\cos(\phi_1(t)))\left(\frac{d}{dt}\theta_2(t)\right)+(-$$

$$L_1^2\cos(\theta_1(t))^2+L_1^2)\left(\frac{d}{dt}\phi_1(t)\right)^2+2\sin(\theta_1(t))\sin(\theta_2(t))\left(\frac{d}{dt}\right.$$

$$\left.\phi_2(t)\right)L_1L_2(\sin(\phi_1(t))\sin(\phi_2(t))+\cos(\phi_2(t))\cos(\phi_1(t)))\left(\frac{d}{dt}\phi_1(t)\right)+($$

$$-\cos(\theta_2(t))^2L_2^2+L_2^2)\left(\frac{d}{dt}\phi_2(t)\right)^2\Bigg)\Bigg)$$

>

> *KE*[1];

$$\frac{m_1L_1^2\left(-\left(\frac{d}{dt}\phi_1(t)\right)^2\cos(\theta_1(t))^2+\left(\frac{d}{dt}\theta_1(t)\right)^2+\left(\frac{d}{dt}\phi_1(t)\right)^2\right)}{2} \qquad \textbf{(12)}$$

> *KE*[2];

$$\frac{1}{2}\Bigg(m_2\Bigg(\left(\frac{d}{dt}\theta_1(t)\right)^2L_1^2+2L_1\Big((\cos(\theta_2(t))(\sin(\phi_1(t))\sin(\phi_2(t)) \qquad \textbf{(13)}$$

$$+\cos(\phi_2(t))\cos(\phi_1(t)))\cos(\theta_1(t))+\sin(\theta_1(t))\sin(\theta_2(t)))\left(\frac{d}{dt}\theta_2(t)\right)$$

$$+\cos(\theta_1(t))\sin(\theta_2(t))\left(\frac{d}{dt}\phi_2(t)\right)(\sin(\phi_1(t))\cos(\phi_2(t))-\sin(\phi_2(t))\cos(\phi_1(t))))$$

$$L_2\left(\frac{d}{dt}\theta_1(t)\right)+\left(\frac{d}{dt}\theta_2(t)\right)^2L_2^2-2\cos(\theta_2(t))\sin(\theta_1(t))\left(\frac{d}{dt}\right.$$

$$\left.\phi_1(t)\right)L_1L_2(\sin(\phi_1(t))\cos(\phi_2(t))-\sin(\phi_2(t))\cos(\phi_1(t)))\left(\frac{d}{dt}\theta_2(t)\right)+(-$$

$$L_1^2\cos(\theta_1(t))^2+L_1^2)\left(\frac{d}{dt}\phi_1(t)\right)^2+2\sin(\theta_1(t))\sin(\theta_2(t))\left(\frac{d}{dt}\right.$$

$$\left.\phi_2(t)\right)L_1L_2(\sin(\phi_1(t))\sin(\phi_2(t))+\cos(\phi_2(t))\cos(\phi_1(t)))\left(\frac{d}{dt}\phi_1(t)\right)+($$

$$-\cos(\theta_2(t))^2L_2^2+L_2^2)\left(\frac{d}{dt}\phi_2(t)\right)^2\Bigg)\Bigg)$$

> $KEs[1] := \frac{m_1 L_1^2 \left(\sin(\theta_1(t))^2 \left(\frac{d}{dt}\phi_1(t)\right)^2 + \left(\frac{d}{dt}\theta_1(t)\right)^2\right)}{2}$

$$KEs_1 := \frac{m_1 L_1^2 \left(\sin(\theta_1(t))^2 \left(\frac{d}{dt}\phi_1(t)\right)^2 + \left(\frac{d}{dt}\theta_1(t)\right)^2\right)}{2} \quad \textbf{(14)}$$

> $simplify(KE[1] - KEs[1]);$

$$0 \quad \textbf{(15)}$$

>

> $Lag := simplify(KEs[1] + KE[2] - PE);$

$$Lag := -\frac{L_1^2 (\cos(\theta_1(t)) - 1)(\cos(\theta_1(t)) + 1)(m_1 + m_2)\left(\frac{d}{dt}\phi_1(t)\right)^2}{2} + \sin(\theta_1(t)) \left(\cos(\theta_2(t)) (\sin(\phi_2(t))\cos(\phi_1(t)) - \sin(\phi_1(t))\cos(\phi_2(t))) \left(\frac{d}{dt}\theta_2(t)\right) + \sin(\theta_2(t)) \left(\frac{d}{dt}\phi_2(t)\right)(\sin(\phi_1(t))\sin(\phi_2(t)) + \cos(\phi_2(t))\cos(\phi_1(t)))\right) m_2 L_2 L_1 \left(\frac{d}{dt}\phi_1(t)\right) + \frac{L_1^2 (m_1 + m_2)\left(\frac{d}{dt}\theta_1(t)\right)^2}{2} + \left(\left(\cos(\theta_2(t))(\sin(\phi_1(t))\sin(\phi_2(t)) + \cos(\phi_2(t))\cos(\phi_1(t)))\cos(\theta_1(t)) + \sin(\theta_1(t))\sin(\theta_2(t))\right)\left(\frac{d}{dt}\theta_2(t)\right) - \cos(\theta_1(t))\sin(\theta_2(t))\left(\frac{d}{dt}\phi_2(t)\right)(\sin(\phi_2(t))\cos(\phi_1(t)) - \sin(\phi_1(t))\cos(\phi_2(t)))\right) m_2 L_2 L_1 \left(\frac{d}{dt}\theta_1(t)\right) + \frac{\left(\frac{d}{dt}\theta_2(t)\right)^2 L_2^2 m_2}{2} + \frac{\left(-\cos(\theta_2(t))^2 L_2^2 m_2 + L_2^2 m_2\right)\left(\frac{d}{dt}\phi_2(t)\right)^2}{2} + g\left(L_1 (m_1 + m_2)\cos(\theta_1(t)) + \cos(\theta_2(t)) L_2 m_2\right) \quad \textbf{(16)}$$

>

> $Lags := \frac{L_1^2 \sin(\theta_1(t))^2 (m_1 + m_2)\left(\frac{d}{dt}\phi_1(t)\right)^2}{2} + L_1 \sin(\theta_1(t)) m_2 L_2 \left(\cos(\theta_2(t))\sin(\phi_2(t) - \phi_1(t))\left(\frac{d}{dt}\theta_2(t)\right) + \sin(\theta_2(t))\left(\frac{d}{dt}\phi_2(t)\right)\cos(\phi_2(t) - \phi_1(t))\right)\left(\frac{d}{dt}\phi_1(t)\right) + \frac{L_1^2 (m_1 + m_2)\left(\frac{d}{dt}\theta_1(t)\right)^2}{2}$

$$+ L_1 \left(\left(\cos(\theta_2(t)) \cos(\phi_2(t) - \phi_1(t)) \cos(\theta_1(t)) + \sin(\theta_1(t)) \sin(\theta_2(t)) \right) \left(\frac{d}{dt} \theta_2(t) \right) - \cos(\theta_1(t)) \sin(\theta_2(t)) \left(\frac{d}{dt} \phi_2(t) \right) \sin(\phi_2(t) - \phi_1(t)) \right) m_2 L_2 \left(\frac{d}{dt} \theta_1(t) \right) + \frac{\left(\frac{d}{dt} \theta_2(t) \right)^2 L_2^2 m_2}{2} + \frac{\left(-\cos(\theta_2(t))^2 L_2^2 m_2 + L_2^2 m_2 \right) \left(\frac{d}{dt} \phi_2(t) \right)^2}{2} + g \left(L_1 (m_1 + m_2) \cos(\theta_1(t)) + \cos(\theta_2(t)) L_2 m_2 \right);$$

$$Lags := \frac{L_1^2 \sin(\theta_1(t))^2 (m_1 + m_2) \left(\frac{d}{dt} \phi_1(t) \right)^2}{2} + L_1 \sin(\theta_1(t)) m_2 L_2 \left(-\cos(\theta_2(t)) \sin(-\phi_2(t) + \phi_1(t)) \left(\frac{d}{dt} \theta_2(t) \right) + \sin(\theta_2(t)) \left(\frac{d}{dt} \phi_2(t) \right) \cos(-\phi_2(t) + \phi_1(t)) \right) \left(\frac{d}{dt} \phi_1(t) \right) + \frac{L_1^2 (m_1 + m_2) \left(\frac{d}{dt} \theta_1(t) \right)^2}{2} + L_1 \left(\left(\cos(\theta_2(t)) \cos(-\phi_2(t) + \phi_1(t)) \cos(\theta_1(t)) + \sin(\theta_1(t)) \sin(\theta_2(t)) \right) \left(\frac{d}{dt} \theta_2(t) \right) + \cos(\theta_1(t)) \sin(\theta_2(t)) \left(\frac{d}{dt} \phi_2(t) \right) \sin(-\phi_2(t) + \phi_1(t)) \right) m_2 L_2 \left(\frac{d}{dt} \theta_1(t) \right) + \frac{\left(\frac{d}{dt} \theta_2(t) \right)^2 L_2^2 m_2}{2} + \frac{\left(-\cos(\theta_2(t))^2 L_2^2 m_2 + L_2^2 m_2 \right) \left(\frac{d}{dt} \phi_2(t) \right)^2}{2} + g \left(L_1 (m_1 + m_2) \cos(\theta_1(t)) + \cos(\theta_2(t)) L_2 m_2 \right) \quad \textbf{(17)}$$

> *simplify*(*Lag* − *Lags*);

$$0 \quad \textbf{(18)}$$

>

Do the Euler-Lagrange calculations.

```
> Lagm := subs({diff(theta[1](t), t) = theta1p}, Lags) :
> Lagm := subs({diff(theta[2](t), t) = theta2p}, Lagm) :
> Lagm := subs({diff(phi[1](t), t) = phi1p}, Lagm) :
> Lagm := subs({diff(phi[2](t), t) = phi2p}, Lagm) :
> Lagm := subs({theta[1](t) = theta[1]}, Lagm) :
> Lagm := subs({theta[2](t) = theta[2]}, Lagm) :
> Lagm := subs({phi[1](t) = phi[1]}, Lagm) :
> Lagm := simplify(subs({phi[2](t) = phi[2]}, Lagm));
```

$$Lagm := L_1 L_2 m_2 \left(\cos(\theta_1) \cos(\theta_2)\, theta1p\, theta2p + \sin(\theta_1) \sin(\theta_2)\, phi1p\, phi2p \right) \cos(-\phi_2 \quad \textbf{(19)}$$

$$+ \phi_1) + L_1 L_2 m_2 (\cos(\theta_1) \sin(\theta_2)\, phi2p\, theta1p - \cos(\theta_2) \sin(\theta_1)\, phi1p\, theta2p) \sin(-\phi_2 + \phi_1) - \frac{phi1p^2 L_1^2 (m_1 + m_2) \cos(\theta_1)^2}{2} + g L_1 (m_1 + m_2) \cos(\theta_1) - \frac{\cos(\theta_2)^2 phi2p^2 L_2^2 m_2}{2} + \cos(\theta_2) g L_2 m_2 + \sin(\theta_1) \sin(\theta_2)\, theta1p\, theta2p\, L_1 L_2 m_2 + \frac{(phi1p^2 + theta1p^2)(m_1 + m_2) L_1^2}{2} + \frac{L_2^2 m_2 (phi2p^2 + theta2p^2)}{2}$$

>

p1 - partial derivative of Lag with respect to angles theta1,theta2,phi1,phi2
p3 - partial derivative of Lag with respect to t derivatives of angles theta1p,theta2p,phi1p,phi2p
p2 - t derivative of p3

d/dt p3 = p1 or p2 - p1 = 0 A system of 4 equations in 4 unknowns.

> $p1[1] := simplify(diff(Lagm, \theta[1]));$

$$p1_1 := -(-L_2 m_2 (-\sin(\theta_1) \cos(\theta_2)\, theta1p\, theta2p + \cos(\theta_1) \sin(\theta_2)\, phi1p\, phi2p) \cos(-\phi_2 + \phi_1) + L_2 m_2 (\cos(\theta_2) \cos(\theta_1)\, phi1p\, theta2p + \sin(\theta_1) \sin(\theta_2)\, phi2p\, theta1p) \sin(-\phi_2 + \phi_1) + (m_1 + m_2)(-phi1p^2 L_1 \cos(\theta_1) + g) \sin(\theta_1) - \cos(\theta_1) \sin(\theta_2)\, theta1p\, theta2p\, L_2 m_2) L_1 \quad (20)$$

> $p3[1] := simplify(diff(Lagm, theta1p));$

$$p3_1 := (L_2 m_2 \cos(\theta_1) \cos(\theta_2)\, theta2p \cos(-\phi_2 + \phi_1) + L_2 m_2 \cos(\theta_1) \sin(\theta_2)\, phi2p \sin(-\phi_2 + \phi_1) + \sin(\theta_1) \sin(\theta_2)\, theta2p\, L_2 m_2 + theta1p\,(m_1 + m_2) L_1) L_1 \quad (21)$$

> $p1[2] := simplify(diff(Lagm, \theta[2]));$

$$p1_2 := -(L_1 (\cos(\theta_1) \sin(\theta_2)\, theta1p\, theta2p - \sin(\theta_1) \cos(\theta_2)\, phi1p\, phi2p) \cos(-\phi_2 + \phi_1) - L_1 (\cos(\theta_1) \cos(\theta_2)\, phi2p\, theta1p + \sin(\theta_2) \sin(\theta_1)\, phi1p\, theta2p) \sin(-\phi_2 + \phi_1) + (-\sin(\theta_1)\, theta1p\, theta2p\, L_1 - phi2p^2 L_2 \sin(\theta_2)) \cos(\theta_2) + \sin(\theta_2) g) m_2 L_2 \quad (22)$$

> $p3[2] := simplify(diff(Lagm, theta2p));$

$$p3_2 := -L_2 m_2 (\sin(\theta_1) \cos(\theta_2) \sin(-\phi_2 + \phi_1)\, phi1p\, L_1 - \cos(\theta_2) \cos(-\phi_2 + \phi_1) \cos(\theta_1)\, theta1p\, L_1 - \sin(\theta_1) \sin(\theta_2)\, theta1p\, L_1 - theta2p\, L_2) \quad (23)$$

> $p1[3] := simplify(diff(Lagm, \phi[1]));$

$$p1_3 := -L_2 ((\cos(\theta_2) \sin(\theta_1)\, phi1p\, theta2p - \cos(\theta_1) \sin(\theta_2)\, phi2p\, theta1p) \cos(-\phi_2 + \phi_1) + \sin(-\phi_2 + \phi_1)(\cos(\theta_1) \cos(\theta_2)\, theta1p\, theta2p + \sin(\theta_1) \sin(\theta_2)\, phi1p\, phi2p)) m_2 L_1 \quad (24)$$

> $p3[3] := simplify(diff(Lagm, phi1p));$

(25)

$$p3_3 := -\left(-L_2\, m_2 \sin(\theta_1)\sin(\theta_2)\, phi2p \cos(-\phi_2+\phi_1) + L_2\, m_2 \cos(\theta_2)\sin(\theta_1)\, theta2p \sin(-\phi_2+\phi_1) + phi1p\, L_1 (\cos(\theta_1)-1)(\cos(\theta_1)+1)(m_1+m_2)\right) L_1 \tag{25}$$

```
> p1[4] := simplify(diff(Lagm, phi[2]));
```

$$p1_4 := L_2\left(\left(\cos(\theta_2)\sin(\theta_1)\, phi1p\, theta2p - \cos(\theta_1)\sin(\theta_2)\, phi2p\, theta1p\right)\cos(-\phi_2+\phi_1) + \sin(-\phi_2+\phi_1)\left(\cos(\theta_1)\cos(\theta_2)\, theta1p\, theta2p + \sin(\theta_1)\sin(\theta_2)\, phi1p\, phi2p\right)\right) m_2\, L_1 \tag{26}$$

```
> p3[4] := simplify(diff(Lagm, phi2p));
```

$$p3_4 := L_2\, m_2 \left(\sin(\theta_1)\cos(-\phi_2+\phi_1)\sin(\theta_2)\, phi1p\, L_1 + \sin(-\phi_2+\phi_1)\sin(\theta_2)\cos(\theta_1)\, theta1p\, L_1 - \cos(\theta_2)^2\, phi2p\, L_2 + phi2p\, L_2\right) \tag{27}$$

```
>
> for k from 1 by 1 to 4 do
    p1[k] := subs({theta[1] = theta[1](t)}, p1[k]) :
    p1[k] := subs({theta[2] = theta[2](t)}, p1[k]) :
    p1[k] := subs({phi[1] = phi[1](t)}, p1[k]) :
    p1[k] := subs({phi[2] = phi[2](t)}, p1[k]) :
    p1[k] := subs({theta1p = theta1p(t)}, p1[k]) :
    p1[k] := subs({theta2p = theta2p(t)}, p1[k]) :
    p1[k] := subs({phi1p = phi1p(t)}, p1[k]) :
    p1[k] := subs({phi2p = phi2p(t)}, p1[k]) :
    p3[k] := subs({theta[1] = theta[1](t)}, p3[k]) :
    p3[k] := subs({theta[2] = theta[2](t)}, p3[k]) :
    p3[k] := subs({phi[1] = phi[1](t)}, p3[k]) :
    p3[k] := subs({phi[2] = phi[2](t)}, p3[k]) :
    p3[k] := subs({theta1p = theta1p(t)}, p3[k]) :
    p3[k] := subs({theta2p = theta2p(t)}, p3[k]) :
    p3[k] := subs({phi1p = phi1p(t)}, p3[k]) :
    p3[k] := subs({phi2p = phi2p(t)}, p3[k]) :
    p2[k] := diff(p3[k], t) :                       # t derivative
   Leqnc[k] := simplify(p1[k] - p2[k]) :            # Lagrange Equation
   Leqnc[k] := subs({diff(theta[1](t), t) = theta1p}, Leqnc[k]);
   Leqnc[k] := subs({diff(theta[2](t), t) = theta2p}, Leqnc[k]);
   Leqnc[k] := subs({diff(phi[1](t), t) = phi1p}, Leqnc[k]);
   Leqnc[k] := subs({diff(phi[2](t), t) = phi2p}, Leqnc[k]);
   Leqnc[k] := subs({diff(theta1p(t), t) = theta1pp}, Leqnc[k]);
   Leqnc[k] := subs({diff(theta2p(t), t) = theta2pp}, Leqnc[k]);
   Leqnc[k] := subs({diff(phi1p(t), t) = phi1pp}, Leqnc[k]);
   Leqnc[k] := subs({diff(phi2p(t), t) = phi2pp}, Leqnc[k]);
   Leqnc[k] := subs({theta[1](t) = theta[1]}, Leqnc[k]);
   Leqnc[k] := subs({theta[2](t) = theta[2]}, Leqnc[k]);
   Leqnc[k] := subs({phi[1](t) = phi[1]}, Leqnc[k]);
   Leqnc[k] := subs({phi[2](t) = phi[2]}, Leqnc[k]);
   Leqnc[k] := subs({theta1p(t) = theta1p}, Leqnc[k]);
   Leqnc[k] := subs({theta2p(t) = theta2p}, Leqnc[k]);
   Leqnc[k] := subs({phi1p(t) = phi1p}, Leqnc[k]);
```

```
  Leqnc[k] := subs({phi2p(t) = phi2p}, Leqnc[k]);
end:
>
> for k from 1 by 1 to 4 do
    Lenqc[k] := simplify(Leqnc[k]); # The 4 equations
  end
```

$$Lenqc_1 := -\Big(\big(\big(-phi2p^2 - theta2p^2\big)\sin(\theta_2) + theta2pp\cos(\theta_2)\big)\cos(\theta_1)\,L_2\,m_2\cos(-\phi_2 + \phi_1) + 2\cos(\theta_1)\,L_2\left(\cos(\theta_2)\,phi2p\,theta2p + \frac{\sin(\theta_2)\,phi2pp}{2}\right)m_2\sin(-\phi_2+\phi_1) - \sin(\theta_1)\,phi1p^2\,L_1\,(m_1+m_2)\cos(\theta_1) + \big(m_2\,theta2pp\,L_2\sin(\theta_2) + m_2\cos(\theta_2)\,theta2p^2\,L_2 + g\,(m_1+m_2)\big)\sin(\theta_1) + theta1pp\,(m_1+m_2)\,L_1\Big)\,L_1$$

$$Lenqc_2 := -\Big(\big(\big(-phi1p^2 - theta1p^2\big)\sin(\theta_1) + \cos(\theta_1)\,theta1pp\big)\,L_1\cos(\theta_2)\cos(-\phi_2+\phi_1) - 2\,L_1\left(\cos(\theta_1)\,phi1p\,theta1p + \frac{\sin(\theta_1)\,phi1pp}{2}\right)\cos(\theta_2)\sin(-\phi_2+\phi_1) - \cos(\theta_2)\,phi2p^2\,L_2\sin(\theta_2) + \sin(\theta_1)\sin(\theta_2)\,theta1pp\,L_1 + \big(\cos(\theta_1)\,theta1p^2\,L_1 + g\big)\sin(\theta_2) + theta2pp\,L_2\Big)\,m_2\,L_2$$

$$Lenqc_3 := -2\Bigg(-\frac{\big(\big(-phi2p^2 - theta2p^2\big)\sin(\theta_2) + theta2pp\cos(\theta_2)\big)\,L_2\,m_2\sin(\theta_1)\sin(-\phi_2+\phi_1)}{2} + \sin(\theta_1)\,L_2\,m_2\left(\cos(\theta_2)\,phi2p\,theta2p + \frac{\sin(\theta_2)\,phi2pp}{2}\right)\cos(-\phi_2+\phi_1) + (m_1 + m_2)\left(\cos(\theta_1)\sin(\theta_1)\,phi1p\,theta1p - \frac{\cos(\theta_1)^2\,phi1pp}{2} + \frac{phi1pp}{2}\right)L_1\Bigg)\,L_1$$

$$Lenqc_4 := m_2\,L_2\Big(L_1\big(\big(phi1p^2 + theta1p^2\big)\sin(\theta_1) - \cos(\theta_1)\,theta1pp\big)\sin(\theta_2)\sin(-\phi_2 + \phi_1) - 2\left(\cos(\theta_1)\,phi1p\,theta1p + \frac{\sin(\theta_1)\,phi1pp}{2}\right)L_1\sin(\theta_2)\cos(-\phi_2+\phi_1) + L_2\big(-2\cos(\theta_2)\sin(\theta_2)\,phi2p\,theta2p + \cos(\theta_2)^2\,phi2pp - phi2pp\big)\Big) \tag{28}$$

```
>
> for k from 1 by 1 to 4 do
```

```
    Leqn[k] := Leqnc[k]:
  end:
> for k from 1 by 1 to 4 do
    Leqn[k] := simplify(Leqnc[k] - coeff(Leqnc[k], theta1pp, 1)·theta1pp - coeff(Leqnc[k],
      theta2pp, 1)·theta2pp - coeff(Leqnc[k], phi1pp, 1)·phi1pp - coeff(Leqnc[k], phi2pp, 1)
      ·phi2pp + A[k, 1]·ans[1] + A[k, 2]·ans[2] + A[k, 3]·ans[3] + A[k, 4]·ans[4]):
  end:
> for k from 1 by 1 to 4 do
    print(k, simplify(Leqn[k]));
  end
```

$$1, \cos(\theta_1)\sin(\theta_2) L_1 L_2 m_2 (phi2p^2 + theta2p^2)\cos(-\phi_2+\phi_1) - 2\cos(\theta_1)\sin(-\phi_2 + \phi_1)\cos(\theta_2)\, phi2p\, theta2p\, L_1 L_2 m_2 + phi1p^2 L_1^2 (m_1+m_2)\cos(\theta_1)\sin(\theta_1) - L_1 (m_2\cos(\theta_2)\, theta2p^2 L_2 + g(m_1+m_2))\sin(\theta_1) + A_{1,2}\, ans_2 + A_{1,3}\, ans_3 + A_{1,4}\, ans_4 + A_{1,1}\, ans_1$$

$$2, \cos(\theta_2)\sin(\theta_1) L_1 L_2 m_2 (phi1p^2 + theta1p^2)\cos(-\phi_2+\phi_1) + 2\cos(\theta_1)\sin(-\phi_2 + \phi_1)\cos(\theta_2)\, phi1p\, theta1p\, L_1 L_2 m_2 + \cos(\theta_2)\, phi2p^2 L_2^2 m_2 \sin(\theta_2) - L_2 m_2 (\cos(\theta_1)\, theta1p^2 L_1 + g)\sin(\theta_2) + A_{2,3}\, ans_3 + A_{2,4}\, ans_4 + A_{2,1}\, ans_1 + A_{2,2}\, ans_2$$

$$3, -\sin(\theta_1)\sin(\theta_2) L_1 L_2 m_2 (phi2p^2 + theta2p^2)\sin(-\phi_2+\phi_1) - 2\cos(-\phi_2 + \phi_1)\sin(\theta_1)\cos(\theta_2)\, phi2p\, theta2p\, L_1 L_2 m_2 - 2\, phi1p\cos(\theta_1)\, theta1p\, L_1^2 (m_1 + m_2)\sin(\theta_1) + A_{3,2}\, ans_2 + A_{3,3}\, ans_3 + A_{3,4}\, ans_4 + A_{3,1}\, ans_1$$

$$4, \sin(\theta_1)\sin(\theta_2) L_1 L_2 m_2 (phi1p^2 + theta1p^2)\sin(-\phi_2+\phi_1) - 2\cos(\theta_1)\sin(\theta_2)\cos(-\phi_2 + \phi_1)\, phi1p\, theta1p\, L_1 L_2 m_2 - 2\sin(\theta_2)\cos(\theta_2)\, phi2p\, theta2p\, L_2^2 m_2 + A_{4,3}\, ans_3 + A_{4,4}\, ans_4 + A_{4,1}\, ans_1 + A_{4,2}\, ans_2 \qquad \textbf{(29)}$$

```
>
```

Solve for the ODEs using the Euler-Lagrange equations using Maple.

```
> answer := solve({Leqn[1], Leqn[2], Leqn[3], Leqn[4]}, {ans[1], ans[2], ans[3], ans[4]}):
>
> for k from 1 by 1 to 4 do
    ODE[k] := simplify(rhs(answer[k])):
  end:
>
```

Build the 4 x 4 matrix A. Calculate its determinant. Check that A is symmetric.

```
> for k from 1 by 1 to 4 do
```

```
    A[k, 1] := simplify(coeff(Leqnc[k], theta1pp, 1));
    A[k, 2] := simplify(coeff(Leqnc[k], theta2pp, 1));
    A[k, 3] := simplify(coeff(Leqnc[k], phi1pp, 1));
    A[k, 4] := simplify(coeff(Leqnc[k], phi2pp, 1));
  end
```

$$A_{1,\,1} := -\left(m_1+m_2\right) L_1^2$$

$$A_{1,\,2} := -L_1 L_2 m_2 \left(\cos(\theta_2)\cos(-\phi_2+\phi_1)\cos(\theta_1)+\sin(\theta_1)\sin(\theta_2)\right)$$

$$A_{1,\,3} := 0$$

$$A_{1,\,4} := -L_2 m_2 \cos(\theta_1)\sin(\theta_2)\sin(-\phi_2+\phi_1) L_1$$

$$A_{2,\,1} := -L_1 L_2 m_2 \left(\cos(\theta_2)\cos(-\phi_2+\phi_1)\cos(\theta_1)+\sin(\theta_1)\sin(\theta_2)\right)$$

$$A_{2,\,2} := -L_2^2 m_2$$

$$A_{2,\,3} := L_1 \cos(\theta_2)\sin(\theta_1)\sin(-\phi_2+\phi_1) L_2 m_2$$

$$A_{2,\,4} := 0$$

$$A_{3,\,1} := 0$$

$$A_{3,\,2} := L_1 \cos(\theta_2)\sin(\theta_1)\sin(-\phi_2+\phi_1) L_2 m_2$$

$$A_{3,\,3} := -\left(m_1+m_2\right) L_1^2 \sin(\theta_1)^2$$

$$A_{3,\,4} := -L_2 m_2 \sin(\theta_1)\sin(\theta_2)\cos(-\phi_2+\phi_1) L_1$$

$$A_{4,\,1} := -L_2 m_2 \cos(\theta_1)\sin(\theta_2)\sin(-\phi_2+\phi_1) L_1$$

$$A_{4,\,2} := 0$$

$$A_{4,\,3} := -L_2 m_2 \sin(\theta_1)\sin(\theta_2)\cos(-\phi_2+\phi_1) L_1$$

$$A_{4,\,4} := -L_2^2 m_2 \sin(\theta_2)^2 \qquad \textbf{(30)}$$

```
> A := [[A[1, 1], A[1, 2], A[1, 3], A[1, 4]], [A[2, 1], A[2, 2], A[2, 3], A[2, 4]], [A[3, 1], A[3,
    2], A[3, 3], A[3, 4]], [A[4, 1], A[4, 2], A[4, 3], A[4, 4]]];
```

$$A := \left[\left[-\left(m_1+m_2\right) L_1^2,\ -L_1 L_2 m_2 \left(\cos(\theta_2)\cos(-\phi_2+\phi_1)\cos(\theta_1)+\sin(\theta_1)\sin(\theta_2)\right),\ 0,\ -L_2 m_2 \cos(\theta_1)\sin(\theta_2)\sin(-\phi_2+\phi_1) L_1\right],\ \left[-L_1 L_2 m_2 \left(\cos(\theta_2)\cos(-\phi_2+\phi_1)\cos(\theta_1)+\sin(\theta_1)\sin(\theta_2)\right),\ -L_2^2 m_2,\ L_1 \cos(\theta_2)\sin(\theta_1)\sin(-\phi_2+\phi_1) L_2 m_2,\ 0\right],\ \left[0,\ L_1 \cos(\theta_2)\sin(\theta_1)\sin(-\phi_2+\phi_1) L_2 m_2,\ -\left(m_1+m_2\right) L_1^2 \sin(\theta_1)^2,\ -L_2 m_2 \sin(\theta_1)\sin(\theta_2)\cos(-\phi_2+\phi_1) L_1\right],\ \left[-L_2 m_2 \cos(\theta_1)\sin(\theta_2)\sin(-\phi_2+\phi_1) L_1,\ 0,\ -L_2 m_2 \sin(\theta_1)\sin(\theta_2)\cos(-\phi_2+\phi_1) L_1,\ -L_2^2 m_2 \sin(\theta_2)^2\right]\right] \qquad \textbf{(31)}$$

```
> matrix(A);
```

$$\left[\left[-\left(m_1+m_2\right) L_1^2,\ -L_1 L_2 m_2 \left(\cos(\theta_2)\cos(-\phi_2+\phi_1)\cos(\theta_1)+\sin(\theta_1)\sin(\theta_2)\right),\ 0, \qquad \textbf{(32)}$$

$$-L_2\, m_2 \cos(\theta_1) \sin(\theta_2) \sin(-\phi_2+\phi_1)\, L_1\big],$$
$$\big[-L_1\, L_2\, m_2 \left(\cos(\theta_2)\cos(-\phi_2+\phi_1)\cos(\theta_1)+\sin(\theta_1)\sin(\theta_2)\right),\ -L_2^2\, m_2,$$
$$L_1 \cos(\theta_2)\sin(\theta_1)\sin(-\phi_2+\phi_1)\, L_2\, m_2,\ 0\big],$$
$$\big[0,\ L_1\cos(\theta_2)\sin(\theta_1)\sin(-\phi_2+\phi_1)\, L_2\, m_2,\ -(m_1+m_2)\, L_1^2 \sin(\theta_1)^2,$$
$$-L_2\, m_2 \sin(\theta_1)\sin(\theta_2)\cos(-\phi_2+\phi_1)\, L_1\big],$$
$$\big[-L_2\, m_2\cos(\theta_1)\sin(\theta_2)\sin(-\phi_2+\phi_1)\, L_1,\ 0,\ -L_2\, m_2\sin(\theta_1)\sin(\theta_2)\cos(-\phi_2$$
$$+\phi_1)\, L_1,\ -L_2^2\, m_2\sin(\theta_2)^2\big]\big]$$

> *dA* := *simplify*(*det*(*A*));

$$dA := -L_2^4 \sin(\theta_2)^2\, m_2^2 \left(m_2 \left(\cos(\theta_1)-1\right)\left(\cos(\theta_1)+1\right)\left(\cos(\theta_2)-1\right)\left(\cos(\theta_2)+1\right)\cos(\right. \quad \textbf{(33)}$$
$$-\phi_2+\phi_1)^2 + 2\cos(\theta_2)\cos(-\phi_2+\phi_1)\cos(\theta_1)\sin(\theta_1)\sin(\theta_2)\, m_2$$
$$\left. + \cos(\theta_2)^2\cos(\theta_1)^2\, m_2 - m_1 - m_2\right) L_1^4 \sin(\theta_1)^2\, m_1$$

> $dAs := -\left(m_2\sin(\theta_1)^2\sin(\theta_2)^2\cos(-\phi_2+\phi_1)^2 + 2\cos(\theta_2)\cos(-\phi_2\right.$
> $\left.+\phi_1)\cos(\theta_1)\sin(\theta_1)\sin(\theta_2)\, m_2 + \cos(\theta_2)^2\cos(\theta_1)^2\, m_2 - m_1 - m_2\right) L_1^4\sin(\theta_1)^2\, m_1\, m_2^2$
> $L_2^4\sin(\theta_2)^2;$

$$dAs := -\left(m_2\sin(\theta_1)^2\sin(\theta_2)^2\cos(-\phi_2+\phi_1)^2 + 2\cos(\theta_2)\cos(-\phi_2\right. \quad \textbf{(34)}$$
$$\left.+\phi_1)\cos(\theta_1)\sin(\theta_1)\sin(\theta_2)\, m_2 + \cos(\theta_2)^2\cos(\theta_1)^2\, m_2 - m_1 - m_2\right) L_1^4\sin(\theta_1)^2\, m_1\, m_2^2$$
$$L_2^4\sin(\theta_2)^2$$

> *simplify*(*dA* − *dAs*);

$$0 \quad \textbf{(35)}$$

>

> *sort*(*collect*(*dAs*, *m*[1]), *m*[1]);

$$\sin(\theta_1)^2\sin(\theta_2)^2 L_1^4 L_2^4 m_2^2 m_1^2 - \left(m_2\sin(\theta_1)^2\sin(\theta_2)^2\cos(-\phi_2+\phi_1)^2 + 2\cos(\theta_2)\cos(-\phi_2\right. \quad \textbf{(36)}$$
$$\left.+\phi_1)\cos(\theta_1)\sin(\theta_1)\sin(\theta_2)\, m_2 + \cos(\theta_2)^2\cos(\theta_1)^2\, m_2 - m_2\right) L_1^4\sin(\theta_1)^2\, m_2^2$$
$$L_2^4\sin(\theta_2)^2\, m_1$$

> *sort*(*collect*(*dAs*, *m*[2]), *m*[2]);

$$-\left(\sin(\theta_1)^2\sin(\theta_2)^2\cos(-\phi_2+\phi_1)^2 + 2\sin(\theta_1)\sin(\theta_2)\cos(-\phi_2+\phi_1)\cos(\theta_2)\cos(\theta_1)\right. \quad \textbf{(37)}$$
$$\left. + \cos(\theta_2)^2\cos(\theta_1)^2 - 1\right) L_1^4\sin(\theta_1)^2\, m_1\, L_2^4\sin(\theta_2)^2\, m_2^3 + \sin(\theta_1)^2\sin(\theta_2)^2 L_1^4 L_2^4 m_1^2 m_2^2$$

>

Check that A is symmetric.

```
> for k from 1 by 1 to 4 do
    for j from k by 1 to 4 do
      print(A[k, j] − A[j, k]) :
  end
  end
```

$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0 \qquad (38)$$

>

Get the B. Remember to put in the minus sign when solving Ax = B.

```
> for k from 1 by 1 to 4 do
    B[k] := Leqnc[k] − A[k, 1]·theta1pp − A[k, 2]·theta2pp − A[k, 3]·phi1pp − A[k, 4]
      ·phi2pp;
  end
```

$$B_1 := L_2\, m_2 \cos(\theta_1) \sin(\theta_2) \sin(-\phi_2 + \phi_1)\, L_1\, phi2pp + (m_1 + m_2)\, L_1^2\, theta1pp + theta2pp\, L_1\, L_2\, m_2 \left(\cos(\theta_2) \cos(-\phi_2 + \phi_1) \cos(\theta_1) + \sin(\theta_1) \sin(\theta_2)\right) - \Big(-L_2\, m_2 \left(theta2p^2 \cos(\theta_1) \sin(\theta_2) + \cos(\theta_1) \sin(\theta_2)\, phi2p^2 - \cos(\theta_1) \cos(\theta_2)\, theta2pp\right) \cos(-\phi_2 + \phi_1) - L_2\, m_2 \left(-2\, theta2p \cos(\theta_1) \cos(\theta_2)\, phi2p - \cos(\theta_1) \sin(\theta_2)\, phi2pp\right) \sin(-\phi_2 + \phi_1) + \cos(\theta_1) \sin(\theta_2)\, theta1p\, theta2p\, L_2\, m_2 + \sin(\theta_1)\, theta2p^2 \cos(\theta_2)\, L_2\, m_2 + theta1pp\, (m_1 + m_2)\, L_1 + \sin(\theta_1) \sin(\theta_2)\, theta2pp\, L_2\, m_2 + \left(-L_1\, phi1p^2\, (m_1 + m_2) \sin(\theta_1) - theta2p \sin(\theta_2)\, L_2\, m_2\, theta1p\right) \cos(\theta_1) + g \sin(\theta_1)\, (m_1 + m_2) \Big)\, L_1$$

$$B_2 := -L_1 \sin(\theta_1) \cos(\theta_2) \sin(-\phi_2 + \phi_1)\, L_2\, m_2\, phi1pp + theta1pp\, L_1\, L_2\, m_2 \left(\cos(\theta_2) \cos(-\phi_2 + \phi_1) \cos(\theta_1) + \sin(\theta_1) \sin(\theta_2)\right) + L_2^2\, m_2\, theta2pp - \Big(L_1 \left(-\cos(\theta_2) \sin(\theta_1)\, phi1p^2 - \cos(\theta_2) \sin(\theta_1)\, theta1p^2 + theta1pp \cos(\theta_1) \cos(\theta_2)\right) \cos(-\phi_2 + \phi_1)$$

$$- L_1 \left(2 \cos(\theta_1) \cos(\theta_2)\, phi1p\, theta1p + phi1pp \cos(\theta_2) \sin(\theta_1)\right) \sin(-\phi_2 + \phi_1)$$
$$+ theta1p^2 \cos(\theta_1) \sin(\theta_2) L_1 + \sin(\theta_1) \cos(\theta_2)\, theta1p\, theta2p\, L_1$$
$$+ \sin(\theta_1) \sin(\theta_2)\, theta1pp\, L_1 + theta2pp\, L_2 + \left(-\sin(\theta_1)\, theta1p\, theta2p\, L_1\right.$$
$$\left. - phi2p^2 L_2 \sin(\theta_2)\right) \cos(\theta_2) + \sin(\theta_2)\, g \Big) L_2 m_2$$

$$B_3 := phi1pp\, (m_1 + m_2) L_1^2 \sin(\theta_1)^2 + L_2 m_2 \sin(\theta_1) \sin(\theta_2) \cos(-\phi_2 + \phi_1) L_1\, phi2pp$$
$$- L_1 \sin(\theta_1) \cos(\theta_2) \sin(-\phi_2 + \phi_1) L_2 m_2\, theta2pp + \Big($$
$$-L_2 m_2 \left(2 \cos(\theta_2) \sin(\theta_1)\, theta2p\, phi2p + phi2pp \sin(\theta_1) \sin(\theta_2)\right) \cos(-\phi_2 + \phi_1)$$
$$+ L_2 m_2 \left(-\sin(\theta_1) \sin(\theta_2)\, phi2p^2 - \sin(\theta_1) \sin(\theta_2)\, theta2p^2\right.$$
$$\left. + theta2pp \cos(\theta_2) \sin(\theta_1)\right) \sin(-\phi_2 + \phi_1) + L_1 (m_1 + m_2) \Big($$
$$-2 \cos(\theta_1) \sin(\theta_1)\, phi1p\, theta1p + phi1pp \left(\cos(\theta_1)^2 - 1\right) \Big) \Big) L_1$$

$$B_4 := L_2 m_2 \sin(\theta_1) \sin(\theta_2) \cos(-\phi_2 + \phi_1) L_1\, phi1pp + phi2pp\, L_2^2 m_2 \sin(\theta_2)^2 \tag{39}$$
$$+ L_2 m_2 \cos(\theta_1) \sin(\theta_2) \sin(-\phi_2 + \phi_1) L_1\, theta1pp$$
$$- \Big(L_1 \left(2 \cos(\theta_1) \sin(\theta_2)\, phi1p\, theta1p + phi1pp \sin(\theta_1) \sin(\theta_2)\right) \cos(-\phi_2 + \phi_1) + L_1 \Big($$
$$-\sin(\theta_1) \sin(\theta_2)\, phi1p^2 - \sin(\theta_1) \sin(\theta_2)\, theta1p^2 + theta1pp \cos(\theta_1) \sin(\theta_2) \Big) \sin(-\phi_2$$
$$+ \phi_1) - L_2 \left(-2 \cos(\theta_2) \sin(\theta_2)\, phi2p\, theta2p + phi2pp \left(\cos(\theta_2)^2 - 1\right)\right) \Big) m_2 L_2$$

\>

\> b := [−simplify(B[1]),−simplify(B[2]),−simplify(B[3]),−simplify(B[4])];

$$b := \Big[\Big(-\cos(\theta_1) \sin(\theta_2) L_2 m_2 \left(phi2p^2 + theta2p^2\right) \cos(-\phi_2 + \phi_1) + 2 \cos(\theta_2) \cos(\theta_1) \sin(\tag{40}$$
$$-\phi_2 + \phi_1)\, phi2p\, theta2p\, L_2 m_2 + \sin(\theta_1) \left(-phi1p^2 L_1 (m_1 + m_2) \cos(\theta_1)\right.$$
$$\left. + m_2 \cos(\theta_2)\, theta2p^2 L_2 + g\, (m_1 + m_2)\right)\Big) L_1, L_2 \Big(-\cos(\theta_2) \sin(\theta_1) L_1 \left(phi1p^2\right.$$
$$\left. + theta1p^2\right) \cos(-\phi_2 + \phi_1) - 2 \cos(\theta_1) \cos(\theta_2) \sin(-\phi_2 + \phi_1)\, phi1p\, theta1p\, L_1$$
$$+ \sin(\theta_2) \left(\cos(\theta_1)\, theta1p^2 L_1 - phi2p^2 L_2 \cos(\theta_2) + g\right)\Big) m_2,$$
$$2 \sin(\theta_1) \left(\frac{\sin(\theta_2) L_2 m_2 \left(phi2p^2 + theta2p^2\right) \sin(-\phi_2 + \phi_1)}{2} + \cos(-\phi_2\right.$$

$+ \phi_1) \cos(\theta_2)\, phi2p\, theta2p\, L_2\, m_2 + phi1p \cos(\theta_1)\, theta1p\, L_1\, (m_1 + m_2) \Big) L_1,$

$-L_2 \sin(\theta_2)\, m_2 \left(\sin(\theta_1)\, L_1\, (phi1p^2 + theta1p^2) \sin(-\phi_2 + \phi_1) - 2\cos(-\phi_2\right.$

$\left. + \phi_1) \cos(\theta_1)\, phi1p\, theta1p\, L_1 - 2\cos(\theta_2)\, phi2p\, theta2p\, L_2 \right)]$

>

Solve the linear system to get the ODEs using Maple.

```
> ans := linsolve(A, b) :
>
```

Now build the matrices to determine the ODEs using Cramer's rule.

```
> An[1] := [[b[1], A[1, 2], A[1, 3], A[1, 4]], [b[2], A[2, 2], A[2, 3], A[2, 4]], [b[3], A[3, 2],
      A[3, 3], A[3, 4]], [b[4], A[4, 2], A[4, 3], A[4, 4]]] :
> An[2] := [[A[1, 1], b[1], A[1, 3], A[1, 4]], [A[2, 1], b[2], A[2, 3], A[2, 4]], [A[3, 1], b[3],
      A[3, 3], A[3, 4]], [A[4, 1], b[4], A[4, 3], A[4, 4]]] :
> An[3] := [[A[1, 1], A[1, 2], b[1], A[1, 4]], [A[2, 1], A[2, 2], b[2], A[2, 4]], [A[3, 1], A[3,
      2], b[3], A[3, 4]], [A[4, 1], A[4, 2], b[4], A[4, 4]]] :
> An[4] := [[A[1, 1], A[1, 2], A[1, 3], b[1]], [A[2, 1], A[2, 2], A[2, 3], b[2]], [A[3, 1], A[3,
      2], A[3, 3], b[3]], [A[4, 1], A[4, 2], A[4, 3], b[4]]] :
>
> for k from 1 by 1 to 4 do
    dAn[k] := det(An[k]) :
  end:
>
> for k from 1 by 1 to 4 do
    CramerODE[k] := simplify( det(An[k]) / det(A) ) :
  end:
>
```

Print out the four ODEs for theta_1",theta_2",phi_1" and phi_2"

```
> simplify(ans[1]);
```

$\Big(\sin(\theta_1)\, m_2\, (\cos(\theta_2) - 1)\, (\cos(\theta_2) + 1)\, (\cos(\theta_1)\, theta1p^2\, L_1 + g) \cos(-\phi_2 + \phi_1)^2 - \big($ **(41)**

$-\cos(\theta_1)\, phi2p^2\, L_2 \cos(\theta_2)^2 + \big(L_1\, (phi1p^2 + 2\, theta1p^2) \cos(\theta_1)^2 + g\cos(\theta_1)$

$- L_1\, (phi1p^2 + theta1p^2)\big) \cos(\theta_2) + \cos(\theta_1)\, L_2\, (phi2p^2 + theta2p^2)\big) \sin(\theta_2)\, m_2 \cos($

$-\phi_2 + \phi_1) + \sin(\theta_1)\, \big(-phi2p^2\, L_2\, m_2 \cos(\theta_2)^3 + \cos(\theta_1)\, L_1\, m_2\, (phi1p^2$

$$+ \mathit{theta1p}^2\big) \cos(\theta_2)^2 + L_2\, m_2\, (\mathit{phi2p}^2 + \mathit{theta2p}^2) \cos(\theta_2) + (m_1 + m_2) \big(-\mathit{phi1p}^2 L_1 \cos(\theta_1) + g\big)\big)\big) \Big/ \Big(\big(m_2 (\cos(\theta_1) - 1) (\cos(\theta_1) + 1) (\cos(\theta_2) - 1) (\cos(\theta_2) + 1) \cos(-\phi_2 + \phi_1)^2 + 2 \cos(\theta_2) \cos(-\phi_2 + \phi_1) \cos(\theta_1) \sin(\theta_1) \sin(\theta_2)\, m_2 + \cos(\theta_2)^2 \cos(\theta_1)^2 m_2 - m_1 - m_2 \big) L_1 \Big)$$

> simplify(ans[2]);

$$\Big(\cos(\theta_2) \sin(\theta_2)\, \mathit{theta2p}^2 L_2\, m_2 (\cos(\theta_1) - 1) (\cos(\theta_1) + 1) \cos(-\phi_2 + \phi_1)^2 - \Big(-\cos(\theta_2)\, \mathit{phi1p}^2 L_1 (m_1 + m_2) \cos(\theta_1)^2 + \big(L_2\, m_2 (\mathit{phi2p}^2 + 2\, \mathit{theta2p}^2) \cos(\theta_2)^2 + g (m_1 + m_2) \cos(\theta_2) - L_2\, m_2 (\mathit{phi2p}^2 + \mathit{theta2p}^2) \big) \cos(\theta_1) + \cos(\theta_2)\, L_1 (\mathit{phi1p}^2 + \mathit{theta1p}^2) (m_1 + m_2) \Big) \sin(\theta_1) \cos(-\phi_2 + \phi_1) + \sin(\theta_2) \Big(-\mathit{phi1p}^2 L_1 (m_1 + m_2) \cos(\theta_1)^3 + \big(L_2\, m_2 (\mathit{phi2p}^2 + \mathit{theta2p}^2) \cos(\theta_2) + g (m_1 + m_2) \big) \cos(\theta_1)^2 + L_1 (\mathit{phi1p}^2 + \mathit{theta1p}^2) (m_1 + m_2) \cos(\theta_1) - \cos(\theta_2)\, \mathit{phi2p}^2 L_2 (m_1 + m_2) \Big) \Big) \Big/ \Big(\big(m_2 (\cos(\theta_1) - 1) (\cos(\theta_1) + 1) (\cos(\theta_2) - 1) (\cos(\theta_2) + 1) \cos(-\phi_2 + \phi_1)^2 + 2 \cos(\theta_2) \cos(-\phi_2 + \phi_1) \cos(\theta_1) \sin(\theta_1) \sin(\theta_2)\, m_2 + \cos(\theta_2)^2 \cos(\theta_1)^2 m_2 - m_1 - m_2 \big) L_2 \Big) \quad \textbf{(42)}$$

> simplify(ans[3]);

$$\Big(-2 \cos(\theta_1)\, \mathit{phi1p}\, \mathit{theta1p}\, L_1\, m_2 (\cos(\theta_2) - 1) (\cos(\theta_2) + 1) (\cos(\theta_1) - 1) (\cos(\theta_1) + 1) \cos(-\phi_2 + \phi_1)^2 - m_2 \Big(\big((\cos(\theta_2) + 1) \big(-\cos(\theta_1)^2 \mathit{phi1p}^2 L_1 + g \cos(\theta_1) + L_1 (\mathit{phi1p}^2 + \mathit{theta1p}^2) \big) (\cos(\theta_2) - 1) \sin(-\phi_2 + \phi_1) + 4 \cos(\theta_1)^2 \cos(\theta_2) \sin(\theta_2)\, \mathit{phi1p}\, \mathit{theta1p}\, L_1 \big) \sin(\theta_1) \cos(-\phi_2 + \phi_1) + \sin(\theta_2)\, m_2 \Big(-\cos(\theta_2)\, \mathit{phi1p}^2 L_1 \cos(\theta_1)^3 + g \cos(\theta_1)^2 \cos(\theta_2) + \cos(\theta_2)\, L_1 (\mathit{phi1p}^2 + \mathit{theta1p}^2) \cos(\theta_1) - L_2 \big(\cos(\theta_2)^2 \mathit{phi2p}^2 - \mathit{phi2p}^2 - \mathit{theta2p}^2 \big) \Big) \sin(-\phi_2 + \phi_1) - 2 \cos(\theta_1)\, \mathit{phi1p}\, \mathit{theta1p}\, L_1 \big(\cos(\theta_2)^2 \cos(\theta_1)^2 m_2 - m_1 - m_2 \big) \Big) \Big/ \Big(\big(m_2 (\cos(\theta_1) - 1) (\cos(\theta_1) + 1) (\cos(\theta_2) - 1) (\cos(\theta_2) + 1) \cos(-\phi_2 + \phi_1)^2 + 2 \cos(\theta_2) \cos(-\phi_2 + \phi_1) \cos(\theta_1) \sin(\theta_1) \sin(\theta_2)\, m_2 + \cos(\theta_2)^2 \cos(\theta_1)^2 m_2 - m_1 - m_2 \big) L_1 \sin(\theta_1) \Big) \quad \textbf{(43)}$$

> simplify(ans[4]);

$$\Big(\Big(-\sin(\theta_2)\, L_2\, m_2 \big(\cos(\theta_2)^2 \mathit{phi2p}^2 - \mathit{phi2p}^2 - \mathit{theta2p}^2 \big) (\cos(\theta_1) - 1) (\cos(\theta_1) + 1) \cos(\quad \textbf{(44)}$$

$$-\phi_2+\phi_1\big) - \Big(-phi1p^2 L_1 (m_1+m_2)\cos(\theta_1)^2 + \Big(-phi2p^2 L_2 m_2 \cos(\theta_2)^3 + L_2 m_2 (phi2p^2 + theta2p^2)\cos(\theta_2) + g(m_1+m_2)\Big)\cos(\theta_1) + L_1 (phi1p^2 + theta1p^2)(m_1+m_2)\Big)\sin(\theta_1)\Big)\sin(-\phi_2+\phi_1) - 2\Big(m_2(\cos(\theta_1)-1)(\cos(\theta_1)+1)(\cos(\theta_2)-1)(\cos(\theta_2)+1)\cos(-\phi_2+\phi_1)^2 + 2\cos(\theta_2)\cos(-\phi_2+\phi_1)\cos(\theta_1)\sin(\theta_1)\sin(\theta_2) m_2 + \cos(\theta_2)^2\cos(\theta_1)^2 m_2 - m_1 - m_2\Big)\cos(\theta_2) L_2\, theta2p\, phi2p\Big) \Big/ \Big(\sin(\theta_2)\Big(m_2(\cos(\theta_1)-1)(\cos(\theta_1)+1)(\cos(\theta_2)-1)(\cos(\theta_2)+1)\cos(-\phi_2+\phi_1)^2 + 2\cos(\theta_2)\cos(-\phi_2+\phi_1)\cos(\theta_1)\sin(\theta_1)\sin(\theta_2) m_2 + \cos(\theta_2)^2\cos(\theta_1)^2 m_2 - m_1 - m_2\Big) L_2\Big)$$

>

> $m[1] := 0;$

$$m_1 := 0 \qquad \textbf{(45)}$$

> $simplify(ans[1]);$

Error, (in simplify/trig) indeterminate expression of the form 0/0

> $simplify(ans[2]);$

Error, (in simplify/trig) indeterminate expression of the form 0/0

> $simplify(ans[3]);$

Error, (in simplify/trig) indeterminate expression of the form 0/0

> $simplify(ans[4]);$

Error, (in simplify/trig) indeterminate expression of the form 0/0

>

> $unassign('m');$

>

> $simplify(limit(ans[1], m[1]=0));$

$$\Big(\sin(\theta_1)(\cos(\theta_2)-1)(\cos(\theta_2)+1)(\cos(\theta_1)\, theta1p^2 L_1 + g)\cos(-\phi_2+\phi_1)^2 - \sin(\theta_2)\Big(-\cos(\theta_1)\, phi2p^2 L_2 \cos(\theta_2)^2 + \Big(L_1(phi1p^2 + 2\, theta1p^2)\cos(\theta_1)^2 + g\cos(\theta_1) - L_1(phi1p^2+theta1p^2)\Big)\cos(\theta_2) + \cos(\theta_1) L_2 (phi2p^2 + theta2p^2)\Big)\cos(-\phi_2+\phi_1) + \Big(-phi2p^2 L_2\cos(\theta_2)^3 + \cos(\theta_1) L_1 (phi1p^2+theta1p^2)\cos(\theta_2)^2 + L_2(phi2p^2+theta2p^2)\cos(\theta_2) - phi1p^2 L_1 \cos(\theta_1) + g\Big)\sin(\theta_1)\Big) \Big/ \Big(L_1\Big(\Big(\big(\cos(\theta_1)^2 - 1\big)\cos(\theta_2)^2 - \cos(\theta_1)^2 + 1\Big)\cos(-\phi_2+\phi_1)^2 + 2\sin(\theta_1)\sin(\theta_2)\cos(-\phi_2+\phi_1)\cos(\theta_2)\cos(\theta_1) + \cos(\theta_2)^2\cos(\theta_1)^2 - 1\Big)\Big) \qquad \textbf{(46)}$$

> $simplify(limit(ans[2], m[1]=0))$

$$\Big(\cos(\theta_2)\sin(\theta_2)\, theta2p^2 L_2 (\cos(\theta_1)-1)(\cos(\theta_1)+1)\cos(-\phi_2+\phi_1)^2 - \sin(\theta_1)\Big(\qquad \textbf{(47)}$$

$$-\cos(\theta_2)\, phi1p^2 L_1 \cos(\theta_1)^2 + \Big(L_2\, (phi2p^2 + 2\, theta2p^2) \cos(\theta_2)^2 + g \cos(\theta_2) - L_2\, (phi2p^2 + theta2p^2)\Big) \cos(\theta_1) + \cos(\theta_2)\, L_1\, (phi1p^2 + theta1p^2)\Big) \cos(-\phi_2 + \phi_1) + \Big(-phi1p^2 L_1 \cos(\theta_1)^3 + \big(L_2\, (phi2p^2 + theta2p^2) \cos(\theta_2) + g\big) \cos(\theta_1)^2 + \cos(\theta_1)\, L_1\, (phi1p^2 + theta1p^2) - phi2p^2 L_2 \cos(\theta_2)\Big) \sin(\theta_2)\Big) \Big/ \Big(L_2 \Big(\Big(\big(\cos(\theta_2)^2 - 1\big) \cos(\theta_1)^2 - \cos(\theta_2)^2 + 1\Big) \cos(-\phi_2 + \phi_1)^2 + 2 \sin(\theta_1) \sin(\theta_2) \cos(-\phi_2 + \phi_1) \cos(\theta_2) \cos(\theta_1) + \cos(\theta_2)^2 \cos(\theta_1)^2 - 1\Big)\Big)$$

> *simplify*(*limit*(*ans*[3], *m*[1] = 0));

$$\Big(-2 \cos(\theta_1)\, phi1p\, theta1p\, L_1\, (\cos(\theta_2) - 1)\, (\cos(\theta_2) + 1)\, (\cos(\theta_1) - 1)\, (\cos(\theta_1) + 1) \cos(-\phi_2 + \phi_1)^2 - \Big(\big(\cos(\theta_2) + 1\big) \Big(-\cos(\theta_1)^2 phi1p^2 L_1 + g \cos(\theta_1) + L_1\, (phi1p^2 + theta1p^2)\Big) (\cos(\theta_2) - 1) \sin(-\phi_2 + \phi_1) + 4 \cos(\theta_1)^2 \cos(\theta_2) \sin(\theta_2)\, phi1p\, theta1p\, L_1\Big) \sin(\theta_1) \cos(-\phi_2 + \phi_1) + \sin(\theta_2) \Big(-\cos(\theta_2)\, phi1p^2 L_1 \cos(\theta_1)^3 + g \cos(\theta_1)^2 \cos(\theta_2) + \cos(\theta_2)\, L_1\, (phi1p^2 + theta1p^2) \cos(\theta_1) - L_2 \big(\cos(\theta_2)^2 phi2p^2 - phi2p^2 - theta2p^2\big)\Big) \sin(-\phi_2 + \phi_1) - 2 \cos(\theta_1)^3 \cos(\theta_2)^2 phi1p\, theta1p\, L_1 + 2 \cos(\theta_1)\, phi1p\, theta1p\, L_1\Big) \Big/ \Big(L_1 \sin(\theta_1) \Big(\Big(\big(\cos(\theta_2)^2 - 1\big) \cos(\theta_1)^2 - \cos(\theta_2)^2 + 1\Big) \cos(-\phi_2 + \phi_1)^2 + 2 \sin(\theta_1) \sin(\theta_2) \cos(-\phi_2 + \phi_1) \cos(\theta_2) \cos(\theta_1) + \cos(\theta_2)^2 \cos(\theta_1)^2 - 1\Big)\Big) \quad \textbf{(48)}$$

> *simplify*(*limit*(*ans*[4], *m*[1] = 0));

$$\Big(-2 \cos(\theta_2)\, phi2p\, theta2p\, L_2\, (\cos(\theta_2) - 1)\, (\cos(\theta_2) + 1)\, (\cos(\theta_1) - 1)\, (\cos(\theta_1) + 1) \cos(-\phi_2 + \phi_1)^2 - \Big(\big(\cos(\theta_2)^2 phi2p^2 - phi2p^2 - theta2p^2\big) (\cos(\theta_1) - 1)\, (\cos(\theta_1) + 1) \sin(-\phi_2 + \phi_1) + 4 \cos(\theta_1) \cos(\theta_2)^2 \sin(\theta_1)\, phi2p\, theta2p\Big) \sin(\theta_2)\, L_2 \cos(-\phi_2 + \phi_1) - \sin(\theta_1) \Big(-\cos(\theta_1)^2 phi1p^2 L_1 + \Big(-phi2p^2 L_2 \cos(\theta_2)^3 + L_2\, (phi2p^2 + theta2p^2) \cos(\theta_2) + g\Big) \cos(\theta_1) + L_1\, (phi1p^2 + theta1p^2)\Big) \sin(-\phi_2 + \phi_1) - 2 \cos(\theta_1)^2 \cos(\theta_2)^3 phi2p\, theta2p\, L_2 + 2 \cos(\theta_2)\, phi2p\, theta2p\, L_2\Big) \Big/ \Big(\sin(\theta_2)\, L_2 \Big(\Big(\big(\cos(\theta_2)^2 - 1\big) \cos(\theta_1)^2 - \cos(\theta_2)^2 + 1\Big) \cos(-\phi_2 + \phi_1)^2 + 2 \sin(\theta_1) \sin(\theta_2) \cos(-\phi_2 + \phi_1) \cos(\theta_2) \cos(\theta_1) + \cos(\theta_2)^2 \cos(\theta_1)^2 - 1\Big)\Big) \quad \textbf{(49)}$$

>

> *m*[2] := 0;

$$m_2 := 0 \tag{50}$$

> *simplify*(*ans*[1]);

$$-\frac{\sin(\theta_1)\left(-phi1p^2 L_1 \cos(\theta_1) + g\right)}{L_1} \tag{51}$$

> *simplify*(*ans*[2]);

$$\frac{1}{L_2}\Big(\cos(\theta_2)\sin(\theta_1)\left(-\cos(\theta_1)^2 phi1p^2 L_1 + g\cos(\theta_1) + L_1\left(phi1p^2 + theta1p^2\right)\right)\cos(-\phi_2 + \phi_1) - \left(-phi1p^2 L_1 \cos(\theta_1)^3 + g\cos(\theta_1)^2 + \cos(\theta_1) L_1\left(phi1p^2 + theta1p^2\right) - phi2p^2 L_2 \cos(\theta_2)\right)\sin(\theta_2)\Big) \tag{52}$$

> *simplify*(*ans*[3]);

$$-\frac{2\cos(\theta_1)\, phi1p\, theta1p}{\sin(\theta_1)} \tag{53}$$

> *simplify*(*ans*[4]);

$$\frac{1}{L_2 \sin(\theta_2)}\Big(\left(-\cos(\theta_1)^2 phi1p^2 L_1 + g\cos(\theta_1) + L_1\left(phi1p^2 + theta1p^2\right)\right)\sin(\theta_1)\sin(-\phi_2 + \phi_1) - 2\cos(\theta_2)\, phi2p\, theta2p\, L_2\Big) \tag{54}$$

>

> *unassign*('*m*');

>

> phi[1] := *ang*; phi[2] := *ang*;

$$\phi_1 := ang$$

$$\phi_2 := ang \tag{55}$$

> *phi1p* := 0; *phi2p* := 0;

$$phi1p := 0$$

$$phi2p := 0 \tag{56}$$

>

> *simplify*(*ans*[1]);

$$\Big(\sin(\theta_1)\, m_2\left(2\cos(\theta_1)\, theta1p^2 L_1 + g\right)\cos(\theta_2)^2 - m_2\left(2\sin(\theta_2)\, theta1p^2 L_1 \cos(\theta_1)^2 - \sin(\theta_1)\, theta2p^2 L_2 - \sin(\theta_2)\, theta1p^2 L_1 + g\cos(\theta_1)\sin(\theta_2)\right)\cos(\theta_2) - m_2\left(\sin(\theta_1)\, theta1p^2 L_1 + \sin(\theta_2)\, theta2p^2 L_2\right)\cos(\theta_1) + g\sin(\theta_1)\, m_1\Big) \Big/ \Big(2 L_1\Big(m_2\left(\cos(\theta_1)^2 - \frac{1}{2}\right)\cos(\theta_2)^2 + \sin(\theta_1)\cos(\theta_1)\cos(\theta_2)\sin(\theta_2)\, m_2 \tag{57}$$

$$-\frac{\cos(\theta_1)^2 m_2}{2} - \frac{m_1}{2}\Big)\Big)$$

> simplify(ans[2]);

$$\Big(\big(2\,m_2\cos(\theta_2)\,theta2p^2\,L_2 + g\,(m_1+m_2)\big)\sin(\theta_2)\cos(\theta_1)^2 + \big(-2\cos(\theta_2)^2\sin(\theta_1)\,theta2p^2\,L_2\,m_2 - g\sin(\theta_1)\,(m_1+m_2)\cos(\theta_2) + \sin(\theta_1)\,theta2p^2\,L_2\,m_2 + \sin(\theta_2)\,theta1p^2\,L_1\,(m_1+m_2)\big)\cos(\theta_1) - \big(theta1p^2\,L_1\,(m_1+m_2)\sin(\theta_1) + \sin(\theta_2)\,theta2p^2\,L_2\,m_2\big)\cos(\theta_2)\Big) \Big/ \Big(2\,L_2\Big(m_2\Big(\cos(\theta_2)^2 - \frac{1}{2}\Big)\cos(\theta_1)^2 + \sin(\theta_1)\cos(\theta_1)\cos(\theta_2)\sin(\theta_2)\,m_2 - \frac{\cos(\theta_2)^2 m_2}{2} - \frac{m_1}{2}\Big)\Big) \quad \textbf{(58)}$$

> simplify(ans[3]);

$$0 \quad \textbf{(59)}$$

> simplify(ans[4]);

$$0 \quad \textbf{(60)}$$

theta_1' = 0 = theta_2', phi_1 = q = phi_2, phi_1' = v = phi_2' Tracing out two cones

>

> theta1p := 0;

$$theta1p := 0 \quad \textbf{(61)}$$

> theta2p := 0;

$$theta2p := 0 \quad \textbf{(62)}$$

> phi[1] := q;

$$\phi_1 := q \quad \textbf{(63)}$$

> phi[2] := q;

$$\phi_2 := q \quad \textbf{(64)}$$

> phi1p := v;

$$phi1p := v \quad \textbf{(65)}$$

> phi2p := v;

$$phi2p := v \quad \textbf{(66)}$$

>

> simplify(ans[1]);

$$\Big(-\sin(\theta_1)\cos(\theta_2)^3 v^2 L_2 m_2 + m_2\big(v^2\,(L_1\sin(\theta_1) + L_2\sin(\theta_2))\cos(\theta_1) + \sin(\theta_1)\,g\big)\cos(\theta_2)^2 - \big(\sin(\theta_2)\,v^2 L_1\cos(\theta_1)^2 + g\cos(\theta_1)\sin(\theta_2) - v^2\,(\sin(\theta_1)\,L_2 + \sin(\theta_2)\,L_1)\big)m_2\cos(\theta_2) - v^2\,\big(L_1\,(m_1+m_2)\sin(\theta_1) + \sin(\theta_2)\,L_2\,m_2\big)\cos(\theta_1) \quad \textbf{(67)}$$

$$+ g \sin(\theta_1)\, m_1\Big) \Big/ \Big(L_1 \big(2 \sin(\theta_1) \cos(\theta_1) \cos(\theta_2) \sin(\theta_2)\, m_2 + 2 \cos(\theta_2)^2 \cos(\theta_1)^2 m_2 - \cos(\theta_1)^2 m_2 - \cos(\theta_2)^2 m_2 - m_1\big)\Big)$$

> *simplify*(*ans*[2]);

$$\Big(-\sin(\theta_2)\, v^2 L_1 (m_1 + m_2) \cos(\theta_1)^3 + \big(v^2 \big(L_1 (m_1 + m_2) \sin(\theta_1) + \sin(\theta_2) L_2 m_2\big) \cos(\theta_2) + g \sin(\theta_2) (m_1 + m_2)\big) \cos(\theta_1)^2 + \big(-\cos(\theta_2)^2 \sin(\theta_1)\, v^2 L_2 m_2 - g \sin(\theta_1) (m_1 + m_2) \cos(\theta_2) + v^2 \big(L_1 (m_1 + m_2) \sin(\theta_2) + \sin(\theta_1) L_2 m_2\big)\big) \cos(\theta_1) - \cos(\theta_2)\, v^2 (m_1 + m_2) \big(L_1 \sin(\theta_1) + L_2 \sin(\theta_2)\big)\Big) \Big/ \Big(L_2 \big(2 \sin(\theta_1) \cos(\theta_1) \cos(\theta_2) \sin(\theta_2)\, m_2 + 2 \cos(\theta_2)^2 \cos(\theta_1)^2 m_2 - \cos(\theta_1)^2 m_2 - \cos(\theta_2)^2 m_2 - m_1\big)\Big) \quad \textbf{(68)}$$

> *simplify*(*ans*[3]);

$$0 \quad \textbf{(69)}$$

> *simplify*(*ans*[4]);

$$0 \quad \textbf{(70)}$$

>

Determine the conditions on L so that theta1 and theta2 are constant and phi_1 and phi_2 are increasing

> *eqn1* := *sort*(*collect*(*numer*(*ans*[1]), *m*[1]), *m*[1]);

$$eqn1 := \big(-\sin(\theta_1) \cos(\theta_1)\, v^2 L_1 + \sin(\theta_1)\, g\big) m_1^2 + \big(\sin(\theta_1)^2 \cos(\theta_2) \sin(\theta_2)\, v^2 L_1 m_2 + \sin(\theta_1) \cos(\theta_1) \cos(\theta_2)^2 v^2 L_1 m_2 + \sin(\theta_1) \cos(\theta_2) \sin(\theta_2)^2 v^2 L_2 m_2 + \cos(\theta_1) \cos(\theta_2)^2 \sin(\theta_2)\, v^2 L_2 m_2 - \sin(\theta_1) \cos(\theta_1)\, v^2 L_1 m_2 - \cos(\theta_1) \sin(\theta_2)\, v^2 L_2 m_2 - \sin(\theta_1) \sin(\theta_2)^2 g\, m_2 - \cos(\theta_1) \cos(\theta_2) \sin(\theta_2)\, g\, m_2 + g \sin(\theta_1)\, m_2\big) m_1 \quad \textbf{(71)}$$

> *eqn2* := *sort*(*collect*(*numer*(*ans*[2]), *m*[1]), *m*[1]);

$$eqn2 := \big(\sin(\theta_1)^2 \cos(\theta_1) \sin(\theta_2)\, v^2 L_1 + \sin(\theta_1) \cos(\theta_1)^2 \cos(\theta_2)\, v^2 L_1 - \sin(\theta_1) \cos(\theta_2)\, v^2 L_1 - \cos(\theta_2) \sin(\theta_2)\, v^2 L_2 - \sin(\theta_1)^2 \sin(\theta_2)\, g - \sin(\theta_1) \cos(\theta_1) \cos(\theta_2)\, g + \sin(\theta_2)\, g\big) m_1^2 + \big(\sin(\theta_1)^2 \cos(\theta_1) \sin(\theta_2)\, v^2 L_1 m_2 + \sin(\theta_1) \cos(\theta_1)^2 \cos(\theta_2)\, v^2 L_1 m_2 + \sin(\theta_1) \cos(\theta_1) \sin(\theta_2)^2 v^2 L_2 m_2 + \cos(\theta_1)^2 \cos(\theta_2) \sin(\theta_2)\, v^2 L_2 m_2 - \sin(\theta_1) \cos(\theta_2)\, v^2 L_1 m_2 - \cos(\theta_2) \sin(\theta_2)\, v^2 L_2 m_2 - \sin(\theta_1)^2 \sin(\theta_2)\, g\, m_2 - \sin(\theta_1) \cos(\theta_1) \cos(\theta_2)\, g\, m_2 + \sin(\theta_2)\, g\, m_2\big) m_1 \quad \textbf{(72)}$$

>

> *simplify*(*eqn1*);

$$\left(-\sin(\theta_1)\cos(\theta_2)^3 v^2 L_2 m_2 + m_2\left(v^2\left(L_1\sin(\theta_1)+L_2\sin(\theta_2)\right)\cos(\theta_1) + \sin(\theta_1)\, g\right)\cos(\theta_2)^2 - \left(\sin(\theta_2)\, v^2 L_1\cos(\theta_1)^2 + g\cos(\theta_1)\sin(\theta_2) - v^2\left(\sin(\theta_1) L_2 + \sin(\theta_2) L_1\right)\right) m_2\cos(\theta_2) - v^2\left(L_1\left(m_1+m_2\right)\sin(\theta_1) + \sin(\theta_2) L_2 m_2\right)\cos(\theta_1) + g\sin(\theta_1)\, m_1\right) m_1 \qquad \textbf{(73)}$$

> *simplify*(*eqn2*);

$$-\left(\sin(\theta_2)\, v^2 L_1\left(m_1+m_2\right)\cos(\theta_1)^3 + \left(-v^2\left(L_1\left(m_1+m_2\right)\sin(\theta_1) + \sin(\theta_2) L_2 m_2\right)\cos(\theta_2) - g\sin(\theta_2)\left(m_1+m_2\right)\right)\cos(\theta_1)^2 + \left(\cos(\theta_2)^2\sin(\theta_1)\, v^2 L_2 m_2 + g\sin(\theta_1)\left(m_1+m_2\right)\cos(\theta_2) - v^2\left(L_1\left(m_1+m_2\right)\sin(\theta_2) + \sin(\theta_1) L_2 m_2\right)\right)\cos(\theta_1) + \cos(\theta_2)\, v^2\left(m_1+m_2\right)\left(L_1\sin(\theta_1)+L_2\sin(\theta_2)\right)\right) m_1 \qquad \textbf{(74)}$$

> *Lans* := *solve*({*eqn1*, *eqn2*}, {*L*[1], *L*[2]});

$$Lans := \left\{L_1 = \frac{\left(\sin(\theta_1)\cos(\theta_2)\, m_1 + \sin(\theta_1)\cos(\theta_2)\, m_2 - \sin(\theta_2)\cos(\theta_1)\, m_2\right) g}{\cos(\theta_1)\sin(\theta_1)\cos(\theta_2)\, v^2 m_1}, L_2 = -\frac{1}{m_1 v^2\cos(\theta_2)\sin(\theta_2)\cos(\theta_1)}\left(\left(\sin(\theta_1)\cos(\theta_2)\, m_1 + \sin(\theta_1)\cos(\theta_2)\, m_2 - \sin(\theta_2)\cos(\theta_1)\, m_1 - \sin(\theta_2)\cos(\theta_1)\, m_2\right) g\right)\right\} \qquad \textbf{(75)}$$

> *simplify*(*Lans*[1]);

$$L_1 = -\frac{g\left(-\sin(\theta_1)\left(m_1+m_2\right)\cos(\theta_2) + \sin(\theta_2)\cos(\theta_1)\, m_2\right)}{\cos(\theta_1)\sin(\theta_1)\cos(\theta_2)\, v^2 m_1} \qquad \textbf{(76)}$$

> *simplify*(*Lans*[2]);

$$L_2 = \frac{g\left(m_1+m_2\right)\left(-\sin(\theta_1)\cos(\theta_2) + \sin(\theta_2)\cos(\theta_1)\right)}{v^2 m_1\cos(\theta_2)\sin(\theta_2)\cos(\theta_1)} \qquad \textbf{(77)}$$

>

> *subs*({*L*[1] = *rhs*(*Lans*[1]), *L*[2] = *rhs*(*Lans*[2])}, *ans*[1]);

$$-\left(\cos(\theta_1)\sin(\theta_1)\cos(\theta_2)\, v^2 m_1\left(\cos(\theta_2)\left(\sin(\theta_1)\cos(\theta_2)\, m_1 + \sin(\theta_1)\cos(\theta_2)\, m_2 - \sin(\theta_2)\cos(\theta_1)\, m_2\right) g\, m_2 - \cos(\theta_2)\left(\sin(\theta_1)\cos(\theta_2)\, m_1 + \sin(\theta_1)\cos(\theta_2)\, m_2 - \sin(\theta_2)\cos(\theta_1)\, m_1 - \sin(\theta_2)\cos(\theta_1)\, m_2\right) g\, m_2\right.\right. \qquad \textbf{(78)}$$

$$+\frac{1}{\cos(\theta_1)}\left(\sin(\theta_2)\sin(\theta_1)\left(\sin(\theta_1)\cos(\theta_2)m_1+\sin(\theta_1)\cos(\theta_2)m_2\right.\right.$$

$$\left.-\sin(\theta_2)\cos(\theta_1)m_2\right)g\,m_2\Big)+\sin(\theta_1)\,g\,m_1^2-\sin(\theta_2)^2\sin(\theta_1)\,g\,m_1m_2+\sin(\theta_1)\,g\,m_1m_2$$

$$-\sin(\theta_2)\cos(\theta_1)\cos(\theta_2)\,g\,m_1m_2$$

$$+\frac{1}{\cos(\theta_2)}\Big(\big(\sin(\theta_1)\cos(\theta_2)m_1+\sin(\theta_1)\cos(\theta_2)m_2-\sin(\theta_2)\cos(\theta_1)m_1$$

$$-\sin(\theta_2)\cos(\theta_1)m_2\big)g\,m_2\Big)$$

$$-\frac{\left(\sin(\theta_1)\cos(\theta_2)m_1+\sin(\theta_1)\cos(\theta_2)m_2-\sin(\theta_2)\cos(\theta_1)m_2\right)g\,m_1}{\cos(\theta_2)}$$

$$-\frac{1}{\cos(\theta_1)}\Big(\sin(\theta_2)\sin(\theta_1)\big(\sin(\theta_1)\cos(\theta_2)m_1+\sin(\theta_1)\cos(\theta_2)m_2$$

$$-\sin(\theta_2)\cos(\theta_1)m_1-\sin(\theta_2)\cos(\theta_1)m_2\big)g\,m_2\Big)$$

$$\left.\left.-\frac{\left(\sin(\theta_1)\cos(\theta_2)m_1+\sin(\theta_1)\cos(\theta_2)m_2-\sin(\theta_2)\cos(\theta_1)m_2\right)g\,m_2}{\cos(\theta_2)}\right)\right)\Big/$$

$$\Big(\big(\sin(\theta_1)\cos(\theta_2)m_1+\sin(\theta_1)\cos(\theta_2)m_2-\sin(\theta_2)\cos(\theta_1)m_2\big)g\,\big($$

$$-\cos(\theta_1)^2\cos(\theta_2)^2m_1m_2-2\sin(\theta_2)\cos(\theta_1)\cos(\theta_2)\sin(\theta_1)m_1m_2$$

$$-\sin(\theta_2)^2\sin(\theta_1)^2m_1m_2+m_1m_2+m_1^2\big)\Big)$$

> *simplify*(*subs*({L[1]=*rhs*(*Lans*[1]), L[2]=*rhs*(*Lans*[2])}, *ans*[1]));

$$0 \qquad \textbf{(79)}$$

> *simplify*(*subs*({L[1]=*rhs*(*Lans*[1]), L[2]=*rhs*(*Lans*[2])}, *ans*[2]));

$$0 \qquad \textbf{(80)}$$

>

> *unassign*('*theta1p*','*theta2p*','phi','*phi1p*','*phi2p*');

>

> *simplify*(*limit*(*ans*[1], theta[1]=0));

$$-\frac{1}{\left(\cos(\theta_2)^2m_2-m_1-m_2\right)L_1}\Big(m_2\sin(\theta_2)\Big(-\cos(\theta_2)^2\,phi2p^2\,L_2+\big(theta1p^2\,L_1 \qquad \textbf{(81)}$$

$$+ g) \cos(\theta_2) + L_2 (phi2p^2 + theta2p^2)) \cos(-\phi_2 + \phi_1))$$

> *simplify*(*limit*(*ans*[2], theta[1] = 0));

$$\frac{(-L_2 (m_1 phi2p^2 - theta2p^2 m_2) \cos(\theta_2) + (m_1 + m_2)(theta1p^2 L_1 + g)) \sin(\theta_2)}{(\cos(\theta_2)^2 m_2 - m_1 - m_2) L_2} \quad \textbf{(82)}$$

> *simplify*(*limit*(*ans*[3], theta[1] = 0));

$$\frac{1}{L_1} \Big(\lim_{\theta_1 \to 0} \Big(m_2 (-(\cos(\theta_2) + 1)(\cos(\theta_2) - 1)(-\cos(\theta_1)^2 phi1p^2 L_1 + g \cos(\theta_1) \quad \textbf{(83)}$$

$$+ L_1 (phi1p^2 + theta1p^2)) \sin(\theta_1) \cos(-\phi_2 + \phi_1) + \sin(\theta_2) ($$

$$-\cos(\theta_2) phi1p^2 L_1 \cos(\theta_1)^3 + g \cos(\theta_1)^2 \cos(\theta_2) + \cos(\theta_2) L_1 (phi1p^2$$

$$+ theta1p^2) \cos(\theta_1) - L_2 (\cos(\theta_2)^2 phi2p^2 - phi2p^2 - theta2p^2))) \sin(-\phi_2 + \phi_1)$$

$$- 2\, phi1p\, L_1 (m_2 (\cos(\theta_1) - 1)(\cos(\theta_1) + 1)(\cos(\theta_2) - 1)(\cos(\theta_2) + 1) \cos($$

$$-\phi_2 + \phi_1)^2 + 2 \cos(\theta_2) \cos(-\phi_2 + \phi_1) \cos(\theta_1) \sin(\theta_1) \sin(\theta_2) m_2$$

$$+ \cos(\theta_2)^2 \cos(\theta_1)^2 m_2 - m_1 - m_2) theta1p \cos(\theta_1) \Big) \Big/ \Big((m_2 (\cos(\theta_1) - 1)(\cos(\theta_1)$$

$$+ 1)(\cos(\theta_2) - 1)(\cos(\theta_2) + 1) \cos(-\phi_2 + \phi_1)^2 + 2 \cos(\theta_2) \cos(-\phi_2$$

$$+ \phi_1) \cos(\theta_1) \sin(\theta_1) \sin(\theta_2) m_2 + \cos(\theta_2)^2 \cos(\theta_1)^2 m_2 - m_1 - m_2) \sin(\theta_1) \Big) \Big)$$

> *simplify*(*limit*(*ans*[4], theta[1] = 0));

$$-\frac{2 \cos(\theta_2)\, phi2p\, theta2p}{\sin(\theta_2)} \quad \textbf{(84)}$$

>

> *simplify*(*limit*(*ans*[1], theta[2] = 0));

$$\frac{(-L_1 (m_1 phi1p^2 - theta1p^2 m_2) \cos(\theta_1) + (theta2p^2 L_2 + g) m_2 + m_1 g) \sin(\theta_1)}{(\cos(\theta_1)^2 m_2 - m_1 - m_2) L_1} \quad \textbf{(85)}$$

> *simplify*(*limit*(*ans*[2], theta[2] = 0));

$$-\frac{1}{(\cos(\theta_1)^2 m_2 - m_1 - m_2) L_2} ((-phi1p^2 L_1 (m_1 + m_2) \cos(\theta_1)^2 + ((theta2p^2 L_2 + g) m_2 \quad \textbf{(86)}$$

$$+ m_1 g) \cos(\theta_1) + L_1 (phi1p^2 + theta1p^2)(m_1 + m_2)) \sin(\theta_1) \cos(-\phi_2 + \phi_1))$$

> *simplify*(*limit*(*ans*[3], theta[2] = 0));

$$-\frac{2 \cos(\theta_1)\, phi1p\, theta1p}{\sin(\theta_1)} \quad \textbf{(87)}$$

> *simplify*(*limit*(*ans*[4], theta[2] = 0));

(88)

$$\frac{1}{L_2}\left(\lim_{\theta_2 \to 0}\left(\left(-\sin(\theta_2)\,L_2\,m_2\left(\cos(\theta_2)^2\,phi2p^2 - phi2p^2 - theta2p^2\right)\left(\cos(\theta_1)-1\right)\left(\cos(\theta_1)+1\right)\cos(-\phi_2+\phi_1) - \left(-phi1p^2\,L_1\,(m_1+m_2)\cos(\theta_1)^2 + \left(-phi2p^2\,L_2\,m_2\cos(\theta_2)^3 + L_2\,m_2\left(phi2p^2+theta2p^2\right)\cos(\theta_2) + g\,(m_1+m_2)\right)\cos(\theta_1) + L_1\left(phi1p^2 + theta1p^2\right)(m_1+m_2)\right)\sin(\theta_1)\right)\sin(-\phi_2+\phi_1) - 2\left(m_2\left(\cos(\theta_1)-1\right)\left(\cos(\theta_1)+1\right)\left(\cos(\theta_2)-1\right)\left(\cos(\theta_2)+1\right)\cos(-\phi_2+\phi_1)^2 + 2\cos(\theta_2)\cos(-\phi_2+\phi_1)\cos(\theta_1)\sin(\theta_1)\sin(\theta_2)\,m_2 + \cos(\theta_2)^2\cos(\theta_1)^2\,m_2 - m_1 - m_2\right)\cos(\theta_2)\,L_2\,theta2p\,phi2p\right) \Big/ \left(\sin(\theta_2)\left(m_2\left(\cos(\theta_1)-1\right)\left(\cos(\theta_1)+1\right)\left(\cos(\theta_2)-1\right)\left(\cos(\theta_2)+1\right)\cos(-\phi_2+\phi_1)^2 + 2\cos(\theta_2)\cos(-\phi_2+\phi_1)\cos(\theta_1)\sin(\theta_1)\sin(\theta_2)\,m_2 + \cos(\theta_2)^2\cos(\theta_1)^2\,m_2 - m_1 - m_2\right)\right)\right) \quad \textbf{(88)}$$

>