Spherical double pendulum vs. planar double pendulum

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(*Excerpts from the extensive research to be published soon*)

The ODEs in angular coordinates

The two ODEs for the planar double pendulum are:

$$\ddot{\theta}_1 = \frac{-(m_2 L_1 \dot{\theta}_1^2 \cos(\theta_1-\theta_2) + m_2 L_2 \dot{\theta}_2^2)\sin(\theta_1-\theta_2) - gm\sin\theta_1 + gm_2\sin\theta_2\cos(\theta_1-\theta_2)}{KL_1} \quad (1)$$

$$\ddot{\theta}_2 = \frac{(mL_1\dot{\theta}_1^2 + m_2L_2\dot{\theta}_2^2\cos(\theta_1-\theta_2))\sin(\theta_1-\theta_2) + gm\sin\theta_1\cos(\theta_1-\theta_2) - gm\sin\theta_2}{KL_2},$$

where $m = m_1 + m_2$, and the kernel $K = m_1 + m_2\sin^2(\theta_1-\theta_2)$.

The four ODEs for the sperical double pendulum obtained with the Maple computer algebra are:

$$\begin{aligned}
\ddot{\theta}_1 &= \frac{\mathrm{Num}_1}{KL_1} \\
\ddot{\theta}_2 &= \frac{\mathrm{Num}_2}{KL_2} \\
\ddot{\varphi}_1 &= \frac{\mathrm{Num}_3}{KL_1\sin\theta_1} \\
\ddot{\varphi}_2 &= \frac{\mathrm{Num}_4}{KL_2\sin\theta_2}
\end{aligned}$$

where

$K = -(m_2\sin^2\theta_1\sin^2\theta_2\cos^2(\varphi_1-\varphi_2) + \frac{1}{2}m_2\sin 2\theta_1\sin 2\theta_2\cos(\varphi_2-\varphi_1) + m_2\cos^2\theta_1\cos^2\theta_2 - m)$

$\mathrm{Num}_1 = m_2\sin\theta_1\sin^2\theta_2(L_1\dot{\theta}_1^2\cos\theta_1+g)\cos^2(\varphi_1-\varphi_2)+(-L_2\dot{\varphi}_2^2\cos\theta_1\cos^2\theta_2+(L_1(\dot{\varphi}_1^2+2\dot{\theta}_1^2)\cos^2\theta_1+g\cos\theta_1-L_1(\dot{\varphi}_1^2+\dot{\theta}_1^2))\cos\theta_2+L_2(\dot{\varphi}_2^2+\dot{\theta}_2^2)\cos\theta_1)m_2\sin\theta_2\cos(\varphi_2-\varphi_1)-\sin\theta_1(-L_2m_2\dot{\varphi}_2^2\cos^3\theta_2+m_2L_1(\dot{\varphi}_1^2+\dot{\theta}_1^2)\cos\theta_1\cos^2\theta_2+L_2m_2(\dot{\varphi}_2^2+\dot{\theta}_2^2)\cos\theta_2+m(-L_1\dot{\varphi}_1^2\cos\theta_1+g))$

$\mathrm{Num}_2 = -(-\frac{1}{2}L_2m_2\dot{\theta}_2^2\sin 2\theta_2\sin^2\theta_1\cos^2(\varphi_1-\varphi_2)-\sin\theta_1(-\dot{\varphi}_1^2L_1m\cos^2\theta_1\cos\theta_2+(L_2m_2(\dot{\varphi}_2^2+2\dot{\theta}_2^2)\cos^2\theta_2+gm\cos\theta_2-L_2m_2(\dot{\varphi}_2^2+\dot{\theta}_2^2))\cos\theta_1+L_1m(\dot{\varphi}_1^2+\dot{\theta}_1^2)\cos\theta_2)\cos(\varphi_2-$

$\varphi_1)+(-L_1 m\dot{\varphi}_1^2\cos^3\theta_1+(L_2 m_2(\dot{\varphi}_2^2+\dot{\theta}_2^2)\cos\theta_2+gm)\cos^2\theta_1+L_1(\dot{\varphi}_1^2+\dot{\theta}_1^2)m\cos\theta_1-L_2 m\dot{\varphi}_2^2\cos\theta_2)\sin\theta_2)$

$\text{Num}_3 = -(m_2(\sin\theta_1\sin^2\theta_2(-L_1\dot{\varphi}_1^2\cos^2\theta_1+g\cos\theta_1+L_1(\dot{\varphi}_1^2+\dot{\theta}_1^2))\cos(\varphi_2-\varphi_1)+\sin\theta_2(-L_1\dot{\varphi}_1^2\cos^3\theta_1\cos\theta_2+g\cos^2\theta_1\cos\theta_2+L_1\cos\theta_1\cos\theta_2(\dot{\varphi}_1^2+\dot{\theta}_1^2)-L_2(\dot{\varphi}_2^2\cos^2\theta_2-(\dot{\varphi}_2^2+\dot{\theta}_2^2))))\sin(\varphi_1-\varphi_2)-2\dot{\theta}_1 L_1(m_2\sin^2\theta_1\sin^2\theta_2\cos^2(\varphi_1-\varphi_2)+\frac{1}{2}m_2\sin 2\theta_1\sin 2\theta_2\cos(\varphi_2-\varphi_1)+m_2\cos^2\theta_1\cos^2\theta_2-m)\dot{\varphi}_1\cos\theta_1)$

$\text{Num}_4 = (-m_2 L_2\sin\theta_2(\dot{\varphi}_2^2\cos^2\theta_2-(\dot{\varphi}_2^2+\dot{\theta}_2^2))\sin^2\theta_1\cos(\varphi_2-\varphi_1)+\sin\theta_1(-L_1\dot{\varphi}_1^2 m\cos^2\theta_1+(-L_2 m_2\dot{\varphi}_2^2\cos^3\theta_2+L_2 m_2(\dot{\varphi}_2^2+\dot{\theta}_2^2)\cos\theta_2+gm)\cos\theta_1+L_1(\dot{\varphi}_1^2+\dot{\theta}_1^2)m))\sin(\varphi_1-\varphi_2)+2\cos\theta_2(m_2\sin^2\theta_1\sin^2\theta_2\cos^2(\varphi_1-\varphi_2)+\frac{1}{2}m_2\sin 2\theta_1\sin 2\theta_2\cos(\varphi_2-\varphi_1)+m_2\cos^2\theta_1\cos^2\theta_2-m)\dot{\theta}_2\dot{\varphi}_2 L_2$

Encoding them in the Taylor Center ODE solver:

K = - (m2*sin2Th1Th2*cosPhi2m1up2 + 0.5*m2*sin2The1*sin2The2*cosPhi2m1 + m2*cos2Th1Th2 - m)

Num1 = m2*sinThe1*sinThe2up2*(L1*cosThe1*dThe1up2 + g)*cosPhi2m1up2 +(-L2*cosThe1*dPhi2up2*cosThe2up2 + (L1*(dPhi1up2 + 2*dThe1up2)*cosThe1up2 + g*cosThe1 - L1*dPh12dTh12)*cosThe2 + L2*cosThe1*dPh22dTh22)*m2*sinThe2*cosPhi2m1 - sinThe1*(-L2*m2*dPhi2up2*cosThe2up3 + m2*L1*cosThe1*dPh12dTh12*cosThe2up2 + L2*m2*dPh22dTh22*cosThe2 + m*(-L1*dPhi1up2*cosThe1 + g))

Num2 = -(-0.5*sin2The2*dThe2up2*L2*m2*sinThe1up2*cosPhi2m1up2 - sinThe1*(-cosThe2*dPhi1up2*L1*m*cosThe1up2 + (L2*m2*(dPhi2up2 + 2*dThe2up2)*cosThe2up2 + g*m*cosThe2 - L2*m2*dPh22dTh22)*cosThe1 + cosThe2*L1*dPh12dTh12*m)*cosPhi2m1 + (-dPhi1up2*L1*m*cosThe1up3 + (L2*m2*dPh22dTh22*cosThe2 + g*m)*cosThe1up2 + L1*dPh12dTh12*m*cosThe1 - cosThe2*dPhi2up2*L2*m)*sinThe2)

Num3 = -(m2*(sinThe1*sinThe2up2*(-L1*dPhi1up2*cosThe1up2 + g*cosThe1 + L1*dPh12dTh12)*cosPhi2m1 + sinThe2*(- L1*cosThe2*dPhi1up2*cosThe1up3 + g*cosThe1up2*cosThe2 + L1*cosThe1*cosThe2*dPh12dTh12 - L2*(cosThe2up2*dPhi2up2 - dPh22dTh22)))*sinPhi1m2 - 2*dThe1*L1*(m2*sin2Th1Th2*cosPhi2m1up2 + 0.5*m2*sin2The1*sin2The2*cosPhi2m1 + m2*cos2Th1Th2 - m)*dPhi1*cosThe1)

Num4 = (-m2*L2*sinThe2*(cosThe2up2*dPhi2up2 - dPh22dTh22)*sinThe1up2*cosPhi2m1 + sinThe1*(-L1*dPhi1up2*m*cosThe1up2 + (-L2*m2*dPhi2up2*cosThe2up3 + L2*m2*dPh22dTh22*cosThe2 + g*m)*cosThe1 + L1*dPh12dTh12*m))*sinPhi1m2 + 2*cosThe2*(m2*sin2Th1Th2*cosPhi2m1up2 + 0.5*sin2The1*sin2The2*cosPhi2m1*m2 + cos2Th1Th2*m2 - m)*dThe2*dPhi2*L2

The ODE is Cartesian coordinates

In the Cartesian coordinates the coordinates of the pointed masses m_1 and m_2 are respectively $(x_1,\ y_1,\ z_1)$ and $(x_2,\ y_2,\ z_2)$ presuming the restraints

$$\begin{aligned} x_1^2 + y_1^2 + z_1^2 &= L_1^2 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 &= L_2^2 \quad \text{or} \quad \Delta x^2 + \Delta y^2 + \Delta z^2 = L_2^2 \\ x_1\dot{x}_1 + y_1\dot{y}_1 + z_1\dot{z}_1 &= 0 \\ \Delta x \Delta\dot{x} + \Delta y \Delta\dot{y} + \Delta z \Delta\dot{z} &= 0 \end{aligned} \tag{2}$$

We consider tensions T_1 and T_2 in the respective rods satisfying two linear equations in T_1 and T_2

$$\begin{aligned} a_{11}T_1 + a_{12}T_2 &= b_1 \\ a_{21}T_1 + a_{22}T_2 &= b_2 \end{aligned} \tag{3}$$

where

$$\begin{aligned} a_{11} &= -\frac{L_1}{m_1}; & a_{12} &= \frac{x_1\Delta x + y_1\Delta y + z_1\Delta z}{m_1 L_2}; \quad b_1 = g z_1 - (\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) \\ a_{21} &= \frac{x_1\Delta x + y_1\Delta y + z_1\Delta z}{m_1 L_1}; & a_{22} &= -L_2\left(\frac{1}{m_1} + \frac{1}{m_2}\right); \quad b_2 = -(\Delta\dot{x}^2 + \Delta\dot{y}^2 + \Delta\dot{z}^2). \end{aligned}$$

The determinant of this system happens to be

$$D = \frac{L_1 L_2}{m_1 m_2} \neq 0$$

The ODEs to integrate are

$$\begin{aligned} m_1\ddot{x}_1 &= -T_1\frac{x_1}{L_1} + T_2\frac{x_2 - x_1}{L_2} \\ m_1\ddot{y}_1 &= -T_1\frac{y_1}{L_1} + T_2\frac{y_2 - y_1}{L_2} \\ m_1\ddot{z}_1 &= -T_1\frac{z_1}{L_1} + T_2\frac{z_2 - z_1}{L_2} - m_1 g \\ m_2\ddot{x}_2 &= -T_2\frac{x_2 - x_1}{L_2} \\ m_2\ddot{y}_2 &= -T_2\frac{y_2 - y_1}{L_2} \\ m_2\ddot{z}_2 &= -T_2\frac{z_2 - z_1}{L_2} - m_2 g \end{aligned} \tag{4}$$

where T_1 and T_2 are known rational functions of $\mathbf{r}_1,\ \mathbf{r}_2,\ \dot{\mathbf{r}}_1,\ \dot{\mathbf{r}}_2$, and the initial values must obey the restraints (2).