1. Unifying view on ODEs and AD: Open questions *(Gofen, 2009).*

It has been proved[1] that the function

$$x(t) = \frac{e^t - 1}{t}, \quad x^{(k)}|_{t=0} = \frac{1}{k+1}, \quad k = 0,\ 1, ... \tag{1.1}$$

(and some others) which satisfies the equations

$$tx' - tx + x - 1 = 0, \quad \text{or} \quad x' = x - \frac{x-1}{t}$$

at the point $t = 0$ cannot satisfy any explicit rational ODE regular at this point

$$x^{(n+1)} = \frac{P(t,\ x,\ x', ...,\ x^{(n)})}{Q(t,\ x,\ x', ...,\ x^{(n)})}, \quad Q|_{t=0} \neq 0,$$

nor indeed any polynomial ODE

$$x^{(n+1)} = P(t,\ x,\ x', ...,\ x^{(n)}). \tag{1.2}$$

At points other than $t = 0$, a rational ODE may be satisfied by $x(t)$, but not the *polynomial* one.

Corollary 1. *Function $x(t)$ can not satisfy explicit polynomial ODE (1.2) at whichever point of the phase space.*

Proof. Suppose $x(t)$ satisfies (1.2) at $t_0 \neq 0$, $x|_{t=t_0} = (e^{t_0} - 1)/t_0$. As an entire function satisfying a regular at all points ODE (1.2), $x(t)$ may be analytically continued from $x|_{t=t_0}$ along any pass to any point satisfying the ODE (1.2) including the point $t = 0$: impossible. □

Although function $x(t)$ cannot satisfy any *one* explicit polynomial ODE, it can satisfy a *system* of such ODEs. Just introduce $y(t) = \frac{1}{t}$, $y' = -y^2$, and then $x(t)$ satisfies a system (say at an initial point $t = 1$)

$$\begin{aligned} x' &= x - xy + y, \qquad & x|_{t=1} &= e - 1 \\ y' &= -y^2, & y|_{t=1} &= 1. \end{aligned}$$

Observe: while $x(t)$ by itself may be analytically continued into the point $t = 0$, for $y(t)$ it is a point of singularity. So this particular system can not contain $x(t)$ at $t = 0$.

What about other systems? Can any explicit polynomial system of ODEs contain $x(t)$ at $t = 0$? This is not yet known. A Problem below, if proved, would establish existence and an example of a new type of special points (besides Poles, Branching, and Essential singularities). In function (1.1) the point $t = 0$ then would be a holomorphic point special in that the property of being elementary is violated.

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[1] Gofen, A., (2008), Unremovable 'Removable' Singularities, *Complex Variables and Elliptic Equations*, Vol. 53, No. 7, p. 633-642.

Flanders, H., (2007), Functions not satisfying implicit polynomial ODE, *J. Differential Equations*, vol. 240, issue 1, September, pp. 164-171.

Problem 1. *Prove that the entire function (1.1) at the point $t = 0$ can not be a solution of an Initial Value Problem (IVP) for any system of m rational ODEs*

$$x' = \frac{P_1(t,\ x,\ y,\ z, ...)}{Q_1(t,\ x,\ y,\ z, ...)} \qquad (1.3)$$
$$y' = \frac{P_2(t,\ x,\ y,\ z, ...)}{Q_2(t,\ x,\ y,\ z, ...)}$$
$$.............$$

whose all denominators $Q_i|_{t=0} \neq 0$, nor indeed can it be a solution of an IVP for any system of explicit polynomial ODEs

$$x' = P_1(t,\ x,\ y,\ z, ...)$$
$$y' = P_2(t,\ x,\ y,\ z, ...)$$
$$.............$$

A more important open statement below represents the gap in the Unifying view (and it would solve this Problem too).

Conjecture 1. *Consider an IVP for a system of rational ODEs (1.3) with nonzero denominators at $t = 0$, so that it has a unique regular solution near $t = 0$, in particular, derivatives $x^{(n)}|_{t=0} = a_n$, $n = 0, 1, 2, ...$ Then there exists an explicit rational ODE of some order $n + 1$*

$$x^{(n+1)} = \frac{F(t,\ x, ...,\ x^{(n)})}{G(t,\ x, ...,\ x^{(n)})}; \qquad x^{(k)}|_{t=0} = a_k,\ k = 1, 2, ...n \qquad (1.4)$$

whose denominator $G|_{t=0} \neq 0$, having $x(t)$ as a unique solution.

The Conjecture claims convertibility of an explicit first order system (1.3) of m rational ODEs into one explicit rational ODE of order $n + 1$. (The opposite conversion is well known and trivial). A rational system of ODEs converts into a polynomial system[2] (with more variables), and further - into special polynomial systems or degree 2, and in squares only. Thus the Conjecture may be rephrased for a polynomial source systems and rational target ODE (1.4) as well. (For the format in squares only it was proved[2] for $m = 2$, yet not for other values of m).

Remark 1. *It is possible to obtain[2] (in many ways) an* implicit *polynomial ODE*

$$P(t,\ x,\ x', ...,\ x^{(n)}) = 0$$

satisfied by any component of the system, say by $x(t)$, but without certainty that
$$\left.\frac{\partial P}{\partial X_n}\right|_{t=0} \neq 0, \quad (X_n = x^{(n)}).$$

Remark 2. *In general, if we do not ask for the rational right hand sides, but allow* arbitrary holomorphic *right hand sides instead, conversion of an explicit system to one explicit ODE (of first order!) is always possible, albeit in a trivial tautological sense. Just consider the solution $x(t)$ of system (1.3), and denote $f(t) = x'(t)$. Then the equation $x' = f(t)$ is the required ODE.*

See a larger context of this Conjecture in this presentation.

[2]Gofen, A., (2009), The Ordinary Differential Equations and Automatic Differentiation Unified, *Complex Variable and Elliptic Equations*, Vol. 54, No. 9, September, 825-854.