THE GAP IN THE UNIFYING VIEW NOT YET CLOSED

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Abstract. This article was written in 2022-2023 when I believed as though I had proved the Conjecture - the gap in the Unifying Theory of the elementary functions. In February 2023, however, I figured out an error in one step of the proof so that the gap is not yet closed. Nevertheless, this text still contributes to the future closure of that gap presenting the steps leading to the goal and discussing the step which failed. This failed step is a New Conjecture awaiting its solution.

In 2009 a new theory in the Complex analysis, unifying the old concept of elementary functions with Ordinary Differential Equations (ODEs), Automatic Differentiation (AD), and analytic continuation, was presented [1]. In it, two competing definitions of the vector elementary vs. scalar (or stand-alone) elementary functions were introduced, so that the question about their equivalency immediately surfaced up, presenting a gap in the Unifying View. This equivalency depended on the Conjecture, claiming that a system of m first-order explicit polynomial ODEs may be converted into one n-order rational ODE with a nonzero denominator at the given point.

Also earlier, in 2007 a similar question emerged in connection with a new type of special points at which the function is holomorphic, but its scalar elementariness is violated [2]. Such a function can satisfy no rational n-order ODE with a nonzero denominator at this point.

Is a similar statement for such points true also for systems of rational ODEs? This question depends on the same Conjecture posed in both papers and remaining unsolved since 2008..

Here is an attempt to prove the Conjecture by its reformulation into terms of algebra, and collaboration with algebraists George Bergman and Alexander Givental, exemplifying a case of successful interdisciplinary cooperation.

Conventions

- All along this paper we deal with holomorphic functions and their derivatives in the complex space. We use the acronyms ODEs for Ordinary Differential Equations, and IVP for Initial Value Problem.
- We are to deal with rational and polynomial *systems* of several first order ODEs, and with *stand alone* n-order ODEs.
- We distinguish *explicit* ODEs such as

$$\begin{aligned} x^{(n)} &= F(t,\ x, ...,\ x^{(n-1)}) \quad \text{or} \\ y' &= G(t,\ x,\ y,\ z, ...) \end{aligned}$$

from *implicit* ODEs

$$P(t,\ x, ...,\ x^{(n)}) = 0$$

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- Speaking about explicit rational ODEs

$$x' = \frac{G(t,\ x,\ y,\ z, ...)}{H(t,\ x,\ y,\ z, ...)} \quad \text{or}$$
$$x^{(n)} = \frac{g(t,\ x, ...,\ x^{(n-1)})}{h(t,\ x, ...,\ x^{(n-1)})}$$

we understand that these rational ODEs are equivalent respectively to implicit polynomial ODEs

$$x'H(t,\ x,\ y,\ z, ...) - G(t,\ x,\ y,\ z, ...) = 0 \quad \text{or}$$
$$x^{(n)}h(t,\ x, ...,\ x^{(n-1)}) - g(t,\ x, ...,\ x^{(n-1)}) = 0$$

which, unlike the rational ones, are defined at all points of the phase space $(t,\ x,\ y,\ z, ...) \in \mathbf{C}^{m+1}$ or $(t,\ x, ...,\ x^{(n-1)}) \in \mathbf{C}^{n+1}$. At that, an implicit polynomial ODE $P(t,\ x, ...,\ x^{(n)}) = 0$ is called singular at a point $(t_0,\ x_0,\ x_1, ...,\ x_n)$ if $\left.\frac{\partial P}{\partial X_n}\right|_{(t_0,\ x_0,\ x_1, ...,\ x_n)} = 0 \quad (\ X_i = x^{(i)})$.
- If the denominators $H(t_0,\ x_0,\ y_0,\ z_0, ...) = 0$ or $h(t_0,\ x_0, ...,\ x_0^{(n-1)}) = 0$ at a certain point, the corresponding explicit rational ODEs are called singular at a the respective points, whose meaning requires a special definition (below).
- When we say that a *holomorphic* at t_0 solution $x(t)$ satisfies a singular at t_0 rational ODEs, this means that $x(t)$ satisfies the respective implicit polynomial ODE.

The Conjecture

Consider a rational system

$$(1) \qquad \begin{aligned} x' &= \frac{g_1(t,\ x,\ y,\ z, ...)}{h_1(t,\ x,\ y,\ z, ...)} \\ y' &= \frac{g_2(t,\ x,\ y,\ z, ...)}{h_2(t,\ x,\ y,\ z, ...)} \\ z' &= \frac{g_3(t,\ x,\ y,\ z, ...)}{h_3(t,\ x,\ y,\ z, ...)} \\ &\cdots\cdots\cdots \end{aligned}$$

and a polynomial system

$$(2) \qquad \begin{aligned} x' &= p_1(t,\ x,\ y,\ z, ...) \\ y' &= p_2(t,\ x,\ y,\ z, ...) \\ z' &= p_3(t,\ x,\ y,\ z, ...) \\ &\cdots\cdots\cdots \end{aligned}$$

referred in the two equivalent forms of the Conjecture below.

Conjecture 1. *For any component, say $x(t)$, of an explicit system in $x(t),\ y(t),\ z(t),...$ of m 1st order...*

... rational ODEs (1) regular at the initial point $(t_0,\ a,\ b,\ c,...)$, ie. all the denominators $h_k\|_{t=t_0} \neq 0$ so that the solution $x(t)$ exists having all the derivatives $x^{(i)}\|_{t=t_0} = a_i,\ i = 0,\ 1,\ 2,...,$	*... polynomial ODEs (2) regular at all points $(t_0,\ a,\ b,\ c,...)$ of its phase space so that at any initial point t_0 the solution $x(t)$ exists having all the derivatives $x^{(i)}\|_{t=t_0} = a_i,\ i = 0,\ 1,\ 2,...,$*

there exists an explicit n-order rational ODE

$$x^{(n)} = \frac{F(t,\ x,\ x',...,\ x^{(n-1)})}{G(t,\ x,\ x',...,\ x^{(n-1)})} \tag{3}$$

or an implicit polynomial ODE

$$Q(t,\ x,\ x',...,\ x^{(n)}) = 0 \tag{4}$$

satisfied by $x(t)$, $x^{(i)}|_{t=t_0} = a_i,\ i = 0,\ 1,\ 2,...,$ both (3) and (4) being regular at the initial point $(t_0,\ a_0,...,\ a_n)$.

Vector vs. scalar elementariness

Here are the main definitions placed into Table 1 for a comparison.

Vector elementariness	*Scalar elementariness*
Definition 1. A function $x(t)$ (as a part of a vector $(x,\ y,\ z,...)$) is called vector-elementary at and near a point $t = t_0$ if it satisfies a rational system (1) regular at $t = t_0$, or polynomial system (2).	**Definition 2**. A function $x(t)$ is called scalar-elementary at and near a point $t = t_0$, if it satisfies an ODE (3) or (4) regular at $t = t_0$.
Definition 3. A holomorphic at $t = t_0$ vector-function $(x,\ y,\ z,...)$ is called *non-elementary, or violating, or losing its vector elementariness* at $t = t_0$ if in its neighborhood (excluding $t = t_0$ itself) vector-function $(x,\ y,\ z,...)$ can satisfy rational system (1) only if (1) is singular at $t = t_0$. A vector-function which cannot satisfy any rational system (1) at all is called non-elementary everywhere. A *singular* at $t = t_0$ vector-function $(x,\ y,\ z,...)$ is considered *non-elementary* at $t = t_0$.	**Definition 4**. A holomorphic at $t = t_0$ function $x(t)$ is called non-elementary, or violating, or losing its scalar elementariness at $t = t_0$ if in its neighborhood (excluding $t = t_0$ itself) function $x(t)$ can satisfy rational ODE (3) only if (3) is singular at $t = t_0$. A function which cannot satisfy any rational ODE (3) is called non-elementary everywhere. A *singular* at $t = t_0$ function $x(t)$ is considered *non-elementary* at $t = t_0$.

Table 1

Remark 1. *Elementariness of a function $x(t)$ at some point t_0 does not mean as though arbitrarily chosen system (1) satisfied by $x(t)$ is necessarily regular at t_0. Any ODE or their system may be intentionally made singular at any point [3].*

That is why an arbitrarily *chosen ODEs satisfied by $x(t)$ may happen to be singular despite elementariness of $x(t)$ at the point. For example, elementary function $x(t) = t^n$ at $t = 0$ satisfies both regular ODE $x' = nt^{n-1}$ and singular ODE $x' = \frac{nx}{t}$.*

Remark 2. *In order to establish elementariness (in either sense) of a function $x(t)$ at a point t_0, it's necessary to produce a system (1) satisfied by $x(t)$ and* regular *at t_0. From t_0, by integrating the ODEs, a property of elementariness may be analytically continued towards any point t where the denominators h_k or G remain nonzero. On the contrary...*

Remark 3. *In order to establish scalar non-elementariness of a function $x(t)$ at a point t_0 it's not enough to merely produce an ODE (1) satisfied by $x(t)$ and singular at t_0: the ODE with a singularity at t_0 may happen to be replaceable with a regular one. It's a challenge to prove non-elementariness of a function. Besides the Euler's Gamma function (non-elementary in either sense at all points), among other functions, so far, only scalar non-elementariness and only for a special kind of functions [2] was proved .*

Explicit polynomial systems (2) do not have singular points in their phase space (though their solution vectors may have singularities). This fact has an interesting implication.

Remark 4. *The concept of* elementariness *(of any kind) of a function $x(t)$ at a point is based on* regularity *of ODEs at the respective point of their phase space. That is why the fact of elementariness may be verified either via* rational *(1) or* polynomial *ODEs (2). However, the opposite concept of* non-elementariness *of a function $x(t)$ is based on singularity of ODEs at the respective point of their phase space. Therefore* non-elementariness *may be expressed only via* rational *ODEs (1) where the denominator disappears at some points of the phase space.*

As any n-order ODE (3) is trivially transformable into a system of n 1st order ODEs (1), the vector elementariness easily follows from the scalar one.

The vice versa statement, however, is more difficult. Transformation of a system (1) of ODEs into one ODE (3) (without the requirement of regularity of (3)) may be achieved via a process of algebraic elimination using resultants (or combinatorially, Appendix in [3]). Both methods are practically difficult. The real challenge, however, was to meet the requirement of *regularity* of the target ODE (3) in the Conjecture.

Now, as the Conjecture is proved, it establishes the equivalence of both definitions of elementariness.

The proof of the Conjecture

The infinite fundamental sequence of polynomial equations. Given an IVP for a polynomial system

$$
\begin{aligned}
x' &= P_1(t,\ x,\ y,\ z, ...), \quad x|_{t=t_0} = a \\
y' &= Q_1(t,\ x,\ y,\ z, ...), \quad y|_{t=t_0} = b \\
z' &= R_1(t,\ x,\ y,\ z, ...), \quad z|_{t=t_0} = c \\
&\dots\dots\dots\dots,
\end{aligned} \tag{5}
$$

we say that an infinite sequence of polynomial equations - the Fundamental Sequence

(6)

$x' = P_1(t,\ x,\ y,\ z,...)$	$y' = Q_1(t,\ x,\ y,\ z,...)$	$z' = ...$	...
... $x^{(k)} = P_k(t,\ x,\ y,\ z,...)$ $x^{(k+1)} = P_{k+1}(t,\ x,\ y,\ z,...)$ ...			

obeying recursive relations

$$\begin{aligned} P_{k+1}(t,\ x,\ y,\ z,...) &= \frac{\partial P_k}{\partial t} + \frac{\partial P_k}{\partial x}x' + \frac{\partial P_k}{\partial y}y' + \frac{\partial P_k}{\partial z}z' + ... \\ &= \frac{\partial P_k}{\partial t} + \frac{\partial P_k}{\partial x}P_1 + \frac{\partial P_k}{\partial y}Q_1 + \frac{\partial P_k}{\partial z}R_1 + ... \end{aligned} \tag{7}$$

corresponds to the ODEs (5) (and the similar infinite sequences correspond also to $y^{(k)},\ z^{(k)},...$ if we needed them).

Remark 5. *Though* $P_{k+1} = \dfrac{dP_k}{dt}$, *similar relations are not true for the partials:* $\dfrac{\partial P_{k+1}}{\partial u} \neq \dfrac{d}{dt}\dfrac{\partial P_k}{\partial u}$, $u = t,\ x,\ y,...,$ *because for the operator* $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + P_1\dfrac{\partial}{\partial x} + Q_1\dfrac{\partial}{\partial y} + R_1\dfrac{\partial}{\partial z} + ...$, $\dfrac{\partial}{\partial u}\dfrac{d}{dt} \neq \dfrac{d}{dt}\dfrac{\partial}{\partial u}$, $u = t,\ x,\ y,...$

Remark 6. *At an initial point* $t = t_0$, *the IVP (5) generates also a sequence of values* $x^{(k)}|_{t=t_0} = a_k,\ k = 1,\ 2,....$ *If only a finite number of* $x^{(k)}|_{t=t_0}$ *are nonzero, then the solution* $x(t)$ *is a polynomial, for which the Conjecture is obviously true. Therefore, for further consideration, we assume that infinitely many* $x^{(k)}|_{t=t_0}$ *are nonzero so that* $\{P_k\}$ *is an infinite sequence of nonzero polynomials*[1].

View all the coefficients at $x',\ y',\ z',...$ of (7) as a matrix in which we consider infinite columns $\left\|\dfrac{\partial P_k}{\partial y}\right\|,\ \left\|\dfrac{\partial P_k}{\partial z}\right\|,...,\ k = 1,\ 2,...$

Remark 7. *Further in this section, we will consider the right-hand sides* P_k *in the fundamental sequence (6) as algebraic (rather than polynomial) functions* holomorphic at the initial point - *because in the process of elimination of undesired variables, we will transform these initially polynomial right-hand sides into algebraic functions regular at the initial point. In other words, we re-write the fundamental sequence (6) in two columns*

(8)

$x' = A_1(t,\ x,\ y,\ z,...)$	$F_1(x';\ t,\ x,\ y,\ z,...) = 0$
...	...
$x^{(k)} = A_k(t,\ x,\ y,\ z,...)$	$F_k(x^{(k)};\ x';...,\ t,\ x,\ y,...) = 0$
$x^{(k+1)} = A_{k+1}(t,\ x,\ y,\ z,...)$	$F_{k+1}(x^{(k+1)};\ x';...,\ t,\ x,\ y,\ ...) = 0$
...	...

where in the process of elimination of $y,\ z,...$ *the former polynomials* P_k *in the first column are transformed into algebraic functions* A_k, *whose algebraicity is verified by*

[1]Unless all the right hand sides in (5) are linear, the degrees of polynomials $\{P_k\}$ grow. In particular, if polynomials in (5) are of degree 2, the degrees of $\{P_k\}$ grow by 1.

the respective polynomials F_k *in the second column regular at* $t = t_0$*, i.e.* $\dfrac{\partial F_k}{\partial X_k} \neq 0$ *at the initial point.*

We will demonstrate existence of an implicit polynomial ODE $F(x^{(n)};\ t,\ x, ...,\ x^{(n-1)}) = 0$ satisfied by $x(t)$ and regular at $t = t_0$ going through the following steps.

(1) **Identifying the invertible equation** for elimination of the *current* undesired variable, say z. Suppose this invertible equation in the sequence (8) is $x^{(i)} = A_i(t,\ x,\ y,\ z, ...)$. The full algebraic function A_i may be a multi-branch function, so that we must take care to choose the branch which satisfies the initial values. After such an equation is identified, the variable z may be (formally) expressed as one of the branches of an algebraic function Z

(9) $$z = Z(t,\ x,\ y,\ x^{(i)},\ x^{(j)}, ...)$$

holomorphic at $t = t_0$. Here again, we must choose the branch satisfied by the given initial values.

(2) **Ridding of all occurrences of the undesired variable** z in the rest of equations (8). Using the formal expression (9), replace z in all the equations of the fundamental sequence (8)

$$x^{(k)} = A_k(t,\ x,\ y,\ Z(...x^{(i)},\ x^{(j)}, ...), ...) = A_k(t,\ x,\ y,\ x^{(i)},\ x^{(j)}, ...)$$

with understanding that for $k = i$ the equation turns into an identity $x^{(i)} = x^{(i)}$. For every algebraic functions A_k there must exist a polynomial $F_k(A_k;\ T,\ X_0,\ Y_0,\ ...X_{k-1})$ verifying that A_k is algebraic.

(3) If the **algebraic functions** A_k **does not have points of branching singularity**, it is, therefore, rational and regular at the given point completing the process of elimination of z. Otherwise, the algebraic function A_k has branches.

(4) **Inspection of every algebraic function** A_k whether its branches have crossing at the initial point $t = t_0$. If they do not, apply the Givental theorem (in the Appendix) to the polynomial equation

(10) $$F_k(A_k;\ T,\ X_0,\ Y_0,\ ...X_{k-1}) = 0, \quad A_k = x^{(k)}, \quad X_i = x^{(i)}$$

According to that Theorem, if there is no self crossing at $t = t_0$, then $\left.\dfrac{\partial F_k}{\partial A_k}\right|_{t=t_0} \neq 0$ so that the polynomial (10) verifies *regularity* of the algebraic functions A_k for the current k. and we can go ahead to verify the regularity for $k+1$. Otherwise if the branches of this A_k cross each other at $t = t_0$...

(5) Choose some subsequent equation in the sequence with number $m > k$ for which the branch A_m does not self cross at $t = t_0$ (we will see that it is possible).

(6) Repeat the above process for elimination of y and other undesired variables.

Step 1 and what remains unknown in it. The goal of Step 1 is to eliminate a current undesired variable, say z, from one of the equations (8) (whose right-hand sides A_k are assumed to be algebraic functions regular at the initial point). In order to do it, we want to identify at least one equation of the infinite sequence (8) for which $\left.\dfrac{\partial A_k}{\partial z}\right|_{t=t_0} \neq 0$ so that this equation be invertible in z. After that, we solve this equation in z formally obtaining an algebraic representation $z = Z(t,\ x,\ \ x',\ y, ...)$ and substitute it into all the remaining equations (8) thus ridding of z. This way we eliminate z from the infinite system (8) diminishing the number of the undesired variable in it.

If we succeed in identifying such an invertible equation for every undesired variables $y,\ z, ...$ in step 1, will fulfill all the steps completing the proof.

What remains unknown in Step 1, however, is what to do in the exceptional case when *all*

$$\left.\frac{\partial A_k}{\partial z}\right|_{t=t_0} = 0\ ,\ k = 1,\ 2, ... \tag{11}$$

in the infinite sequence (8) so that it is impossible to find in it even one equation invertible in z. Such an exceptional situation does exist as we have found several such examples, and extensively studied this situation in the report [5], figuring out interesting properties of the solution of the source system (5) under such exceptional conditions.

In the examples that we have found, it was possible to diminish the number of the undesired variables - and therefore to fulfill the goal the process of elimination even in this exceptional case. However generally we do not know how to diminish the number of the undesired variables in the situation of the "zero column" (11).

Consequently, the following proof presumes that the exceptional situation (11) at step 1 does not happen leaving the Conjecture still open in the case of the "zero column" (11).

Step 2. Now, having the algebraic function (9) *holomorphic* at $t = t_0$, we may replace all occurrences of variable z in all equation of the fundamental sequence (6)

$$x^{(k)} = A_k(t,\ x,\ y,\ Z(...x^{(i)},\ x^{(j)}, ...), ...) = B_k(t,\ x,\ y,\ x^{(i)},\ x^{(j)}, ...) \tag{12}$$

where every B_k is an algebraic function so that infinite sequence of polynomial equations (6) turns into a sequence of algebraic (rather than polynomial) equations

$$x^{(k)} = B_k(t,\ x,\ y,\ x^{(i)},\ x^{(j)} ...)$$

being a superposition of algebraic functions.

Remark 8. *After the substitution (12), all functions A_k in the sequence (8) ought to be replaced with B_k. However, in order to not complicate the notation, we leave in (8) notation A_k assuming that the substitution (12) was already made.*

For each algebraic A_k there exists the polynomial

$$F_k(A_k;\ T,\ X_0,\ Y_0,\ X_i,\ X_j ...) \tag{13}$$

verifying that A_k is algebraic. We are moving to the goal of assuring that

$$\left.\frac{\partial F_k}{\partial A_k}\right|_{t=t_0} \neq 0. \tag{14}$$

Step 3. We must inspect all functions $A_k,\ \ k = 1,\ 2, ...$ If current A_k does not have points of branching singularity, A_k therefore is rational and regular at the given point. Otherwise the algebraic functions A_k have branching singularities.

Step 4. Verify polynomials F_k in (13) at a small vicinity U of the initial point $(a_k;\ t_0,\ a_0,\ y_0,\ a_i,\ a_j)$ if they meet the terms of the Proposition proved by Givental (in the Appendix 1) thus guarantying regularity (14). In order to meet those terms, we must check if the solution A_k of the polynomial equation $F_k = 0$ (13) is *unique* in a small vicinity U in the following sense.

Every algebraic function A_k (12) is viewed as a branch of the full algebraic function defined by the polynomial (13) so that it is the point $(a_k;\ t_0,\ a_0,\ y_0,\ a_i,\ a_j)$ which identifies a particular branch (from many) which is the solution of the source ODEs (5). Analytical continuation of the branch A_k may lead to other branches, some of which (say $\tilde{A}_k$) may pass through the same initial point $(a_k;\ t_0,\ a_0,\ y_0,\ a_i,\ a_j)$. If this is the case, we have two (or more) branches crossing at the initial point. The following example[2] demonstrates this case.

Example 1. *Consider one branch* $y_+ = +x\sqrt{x+1}$ *at the origin (Fig 1). Its only singularity is the branching point at the* $x = -1$. *The other branch is* $y_- = -x\sqrt{x+1}$ *and they both cross at the origin. The polynomial verifying that* y_+ *and* y_- *are branches of the full algebraic function is* $F = y^2 - x^2 - x^3$. *At the origin* $\dfrac{\partial F}{\partial y} = 0$ *(otherwise it would be violation of the theorem of uniqueness and existence of the solution).*

First, assume that A_k is the only branch in a small vicinity U of the initial point $(a_k;\ t_0,\ a_0,\ y_0,\ a_i,\ a_j)$. With such an assumption, we do meet the requirements of the Proposition so that the polynomial equation $F_k = 0$ meets the requirement of regularity (14). Otherwise...

Step 5. The algebraic branches cross each other at $(a_k;\ t_0,\ a_0,\ y_0,\ a_i,\ a_j)$ so that $A_k|_{t=t_0} = \tilde{A}_k|_{t=t_0}$. In the Example 1, the branches cross at nonzero angle so that though $y_+(0) = y_-(0)$, $y'_+(0) \neq y'_-(0)$. If the Example 1 is modified so that say $y_+ = +x^3\sqrt{x+1}$, then $y_+^{(k)}(0) = y_-^{(k)}(0)$ for $k = 0,\ 1,\ 2$, but $y_+^{(3)}(0) \neq y_-^{(3)}(0)$ - Fig 2.

In a general case, let's consider $x^{(k)} = A_k$ vs. another branch $\tilde{x}^{(k)} = \tilde{A}_k$. Assume that $x^{(k+n)}|_{t=t_0} = \tilde{x}^{(k+n)}|_{t=t_0}$ for $n = 0,\ 1, ...$ until infinity. If it were so, that would mean that both branches are identical: $x^{(k)} \equiv \tilde{x}^{(k)}$ contradicting that they are different branches. Therefore, for certain n, $x^{(k+n)}|_{t=t_0} \neq \tilde{x}^{(k+n)}|_{t=t_0}$.

With that in mind, the equations of (12) with numbers $k,\ k+1, ...,\ k+n-1$ replace with the identities, and consider the equation $x^{(k+n)} = A_{k+n}$ together with its verifying polynomial F_{k+n}, which does meet the requirements of the Proposition 1 in the Appendix 1.

Move on inspecting $A_{k+n+1},\ A_{k+n+2},$ until all of them are either regular at the initial point, or replaced with identities.

Now the elimination of z is completed.

Step 6. Repeat the above elimination process of y and possibly other undesired variables. When there remains no undesired variables in the fundamental sequence (8), for any ODE $x^{(k)} = ...$ which is not an identity, the implicit polynomial ODE $F_k = 0$ will be the target n-order ODE claimed by the Conjecture and proving it - though under the limitation that we do not deal with the exceptional case.■

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[2]Courtesy of George Bergman

The remaining part of this report was written when we believed as though the Conjecture were proven "entirely". Hoping, that some days it will happen, here are a few more facts and discussion outlining the future state when the gap in the theory of elementary functions is closed, and how the Unifying theory then will look like.

Remark 9. *The method of this proof is of the "pure existence" type. However, it shows that the process of obtaining the target ODE* $Q = 0$ *(4) generally depends on the initial values* $(t_0,\ x_0,\ y_0,\ z_0, ...)$ *of the source (2) or (1). The critical factor* $\dfrac{\partial Q(T,\ X_0, ...,\ X_n)}{\partial X_n}$ *of the target (4), if expressed via* $(x,\ y,\ z, ...)$ *using the fundamental sequence (6), is*

$$\frac{\partial Q(T,\ X_0, ...,\ X_n)}{\partial X_n} = q(t,\ x,\ y,\ z, ...).$$

Here q *is a polynomial (rather than a constant). As such, it disappears on a non-empty manifold* $q(t,\ x,\ y,\ z, ...) = 0$*. Therefore, when the target ODE (4) is obtained for one point* $(t_0,\ x_0,\ y_0,\ z_0)$ *at which* $q(t_0,\ x_0,\ y_0,\ z_0) \neq 0$*, at other points* $(t,\ x,\ y,\ z, ...)$ *it is possible that* $q(t,\ x,\ y,\ z, ...) = 0$*. Therefore, generally the target* n*-order ODE cannot be one and the same for all points of the phase space of the source system.*

Corollary 1. *Let* t_0 *in a holomorphic function* $x(t)$ *be an isolated point of violation of elementariness meaning that* $x(t)$ *can satisfy no regular rational system (1) or an ODE (3) at* t_0*. At a regular point* $t_1 \neq t_0$ *however, (1) or (3) may be transformed into a polynomial system (2) satisfied by* $x(t)$*. Then, at least one of the components* $y,\ z, ...$ *of (2) must have a singularity at the point* t_0 *so that* t_0 *be unreachable in a process of integration of the system (2) of ODEs from* t_1 *to* t_0*.*

Proof. If we assume that the initial values $x(t_0)$ (with the corresponding initial values $y(t_0),\ z(t_0), ...$) in the IVP for the poly system (2) produce the solution $x(t)$ while it is given that $x(t)$ can satisfy no regular rational ODE (3) at t_0, this would contradicts the claim of the already proven Conjecture that, at any point $(t_0,\ x_0,\ y_0,\ z_0)$ of the phase space of (2) there exist a regular target ODE (3). □

Example 2. *The solution of the rational ODE*

$$x' = x - \frac{x-1}{t}$$

is $x(t) = \dfrac{e^t - 1}{t},\ x(0) = 1$ *losing elementariness at* $t = 0$*. This* $x(t)$ *also satisfies an IVP, say at* $t = 1$*, for the polynomial system,*

$$\begin{aligned} x' &= x - xy + y, & x|_{t=1} &= e - 1 \\ y' &= -y^2, & y|_{t=1} &= 1 \end{aligned}$$

(here $y = \dfrac{1}{t}$*). Indeed, the point* $t = 0$ *is unreachable in this system.*

Discussion

Before the equivalence between the vector- and scalar elementariness was established in this paper, the following comparison Table 2 had been compiled in [3] in order to emphasize the different roles played by these definitions.

Scalar elementariness...	**Vector elementariness...**
	Follows from the scalar elementariness, but the vice versa statement is not yet proved.
The proofs of the fundamental properties *not known*	The fundamental properties **proved**: the closeness of the class of vector-elementary functions in respect to the superposition and the inverse vector-functions, and the fundamental transforms [1].
Not known if contains all Liouville elementary functions	**Contains** the conventional Liouville elementary functions
Is **lost** at isolated points in some functions like $\dfrac{e^t-1}{t}$ or $\dfrac{\sin t}{t}$ at $t=0$.	*Not known* if it is lost at these points
The Γ function and Γ integral are non-elementary at all points in any sense thanks to the Hölder theorem.	

Table 2.

As it shows, the properties which were proved for one type of elementariness, were not proved for the other. Therefore, now all properties in both columns are established and apply to the unified elementariness.

In particular, the answer to the open question posed in [2] whether the function $x(t) = \dfrac{e^t-1}{t}$ at $t=0$ violates also the vector elementariness, is Yes. The same applies to all functions violating scalar elementariness in the Table 1 in [2].

What about modifications of the Conjecture. We have already mentioned earlier, that the Conjecture with some "minor" modifications remains true and easily provable.

For example, if we drop the requirement that the denominator in the target ODE (3) be nonzero, conversion of a polynomial system of ODEs into one n-order ODE (3) is possible either by the method of algebraic elimination with the resultants [2], or via combinatorial approach [3].

Or, if we drop the requirement that the target ODE (3) be rational allowing its right-hand side to be some holomorphic function, conversion of a polynomial system (or ODEs with holomorphic right-hand sides) into one n-order ODE with a holomorphic right-hand side becomes a trivial tautological exercise with a *first order* target ODE $x' = \varphi_1(t)$, where $x = \varphi(t)$ is the component of the solution of the system and $\varphi_1(t) = \varphi'(t)$, as noted in [2].

Here is, however, a modification which would make the Conjecture false.

Remark 10. *If the Conjecture claimed...*

- *that the target ODE (3), instead of being explicit* rational*, were explicit* polynomial *(15); or...*
- *that the target ODE (4), instead of being* implicit *polynomial were* explicit *polynomial (15)*

$$x^{(n)} = Q(t,\ x, ...,\ x^{(n-1)}), \tag{15}$$

so modified Conjecture would be false. The counterexample is a function having points of violation of elementariness (such as $x(t) = \dfrac{e^t - 1}{t}$ *at* $t = 0$*) which cannot satisfy ODE (15) at either points of its phase space. Only elementary functions not having points of violation of elementariness can satisfy (15): for example* $x(t) = e^t$*.*

In his paper [4] of 2022, also C. Kiselman considered some modifications of the Conjecture and of definitions of elementariness, and he made a comprehensive review of history and origin of the concept of elementary functions. However, he quoted this original Conjecture inaccurately, believing as though the order n of the target ODE were the the number m of ODEs in the system plus 1.

Regularization of an ODE. A function $x(t)$ holomorphic at some point t_0 may satisfy many different ODEs: either regular or singular at this point.

Let a holomorphic function $x(t)$ satisfy an implicit poly ODE

$$P(t,\ x,\ x',\ ...,\ x^{(m)}) = 0 \tag{16}$$

which happened to be singular at t_0. Is it always possible to replace the ODE $P = 0$ with another implicit poly ODE

$$Q(t,\ x,\ x',\ ...,\ x^{(n)}) = 0$$

regular at t_0?

The answer depends not on that arbitrary ODE chosen to verify whether a function $x(t)$ is elementary at t_0, but on the fact whether t_0 is a point of violation of elementariness of $x(t)$. If t_0 is not such a point, singular ODE (16) may be replaced with an ODE regular at t_0 (though we do not know how to find it).

If $x(t)$ does not have points of lost elementariness at all, ODE $P = 0$ may be replaced with an *explicit* poly ODE $x^{(n)} = p(t,\ x,\ x',\ ...,\ x^{(n-1)})$.

Here are examples of seemingly quite similar ODEs.

Example 3. *The (entire) function* $x(t) = te^t$ *satisfies the ODE*

$$P = tx' - tx - x = 0$$

singular at $t = 0$*. However* $x(t)$ *satisfies also the ODE*

$$Q = x'' - 2x' + x = 0$$

regular at $t = 0$*. This function* $x(t)$ *does not have points of violated elementariness.*

Example 4. *The (entire) function* $x(t) = \dfrac{e^t - 1}{t},\ x(0) = 1,$ *satisfies ODE*

$$P = tx' - tx + x - 1 = 0$$

singular at $t = 0$*. However, there can not exist a polynomial ODE satisfied by this function and regular at* $t = 0$ *- because this* $x(t)$ *has a point of violated elementariness at* $t = 0$*.*

Taylor evaluation at a point of lost elementariness. When a function $x(t)$ violates elementariness at a point $t = t_0$ while being holomorphic at it, computation of its derivatives $x^{(k)}|_{t=t_0}$ by applying the formulas of the optimized Taylor method (or AD) [1] are no more possible because of singularity of the respective ODEs.

However, these derivatives $x^{(k)}|_{t=t_0}$ may still be evaluated in a special process (outlined in [1]) for obtaining *formal* derivatives of ODE (16) $P = 0$ singular at $t = t_0$. Unlike such an evaluation in the case of regularity, the process of evaluation of a singular ODE generally may evolve into a branching procedure delivering either multiple formal Taylor expansions (convergent or not), or none of them.

However, in particular cases of functions $x(t)$ holomorphic at the point yet violating elementariness at it, this process may generate recursive formulas for $x^{(k)}|_{t=t_0}$ and the convergent Taylor expansions (as considered in [1]).

Example 5. *At $t = 0$, for the function*

$$x(t) = \frac{e^t - 1}{t}, \qquad x(0) = 1$$

satisfying the singular IVP

$$P = tx' - tx + x - 1 = 0$$

the process of formal differentiation yields the true *derivatives and the* convergent *Taylor series:*

$$\begin{aligned} \left.\frac{d^nP}{dt^n}\right|_{t=0} &= \left[tx^{(n+1)} + nx^{(n)} - tx^{(n)} - nx^{(n-1)} + x^{(n)}\right]_{t=0} = \\ &= (n+1)x^{(n)} - nx^{(n-1)} = 0; \\ \left.x^{(n)}\right|_{t=0} &= \frac{1}{n+1} \end{aligned}$$

so that $x(t)$ is the unique solution of this singular IVP, and a regular ODE at $t = 0$ is impossible for this $x(t)$.

Example 6. *The same is true for the IVP*

$$\begin{aligned} P &= tx'' - x = 0; \qquad x(0) = 0, \qquad x'(0) = 1; \\ \left.\frac{d^nP}{dt^n}\right|_{t=0} &= \left[tx^{(n+2)} + nx^{(n+1)} - x^{(n)}\right]_{t=0} = Q_{n+1} = \\ &= nx^{(n+1)} - x^{(n)} = 0; \\ \left.x^{(n)}\right|_{t=0} &= \frac{1}{(n-1)!}, \qquad n \geq 1; \end{aligned}$$

These recursive formulas are special, differing from the general formulas of the optimized differentiation applied to ODEs in the canonical format in [1].

Suggestions on the terminology. As the gap in the Unifying view [1] is closed now so that we can deal with the unified concept of elementary functions, it's the time to fix the terminology concerning elementary functions and the Taylor method framed within the Unifying View, which unifies...

- Elementary functions;
- ODEs as an instrument for evaluation of the derivatives of the solution;
- The class of elementary ODEs closed with regard to their solutions;

- The transformations of all elementary ODEs into the special formats, **making possible...**
  - Optimized computation of n-order derivatives via the canonical systems, **making possible...**
    - The optimized Taylor integration method as a tool of analytic continuation, **revealing...**
      - Special points unreachable via Taylor integration of ODEs, i.e. the points where the solution is holomorphic, but **loses elementariness.**

The seed of the theory discussed here originates in one chapter of the book by Ramon Moore "Interval Analysis" (1966) where he linked the optimized n-order differentiation of general expressions with the elementary functions.

In terms of the Unifying view, the "Moore's elementary functions" must be called "vector elementary". Moore emphasized, that defined by him elementary functions include all those conventional elementary functions called so by Liouville. Therefore, Moore didn't introduce a new term, believing that the same term "elementary functions" must be used also to his generalization of this notion. Moore's generalization merely added a fundamental property to the notion of elementary functions which prior to him were merely a conventional list. I support the Moore's approach. My preferential list of the terms for the unified concept of elementary functions is the following.

(1) "Elementary functions" (no adjectives). When it is necessary to distinguish those of Liouville, or to emphasize scalar- or vector-elementariness, it should be said so.
(2) "Special elementary functions" for those of Liouville vs. "General elementary" for those in the Unifying view.
(3) "Elementary functions" (no adjectives) for those of Liouville. "General elementary" for those in the Unifying view.

Similarly, Moore used the generic term "Taylor method" for the "Optimized Taylor Method", which now is a part of the Unifying view too. I suggest using the term "Optimized Taylor Method", or even dropping the word "Optimized".

Later, the practitioners attached to this method the name Automatic Differentiation (AD) because AD relies on the formulas of the Optimized n-order differentiation of expressions by Moore. However, the term AD in essence denotes algorithms converting the code computing the functions into a code computing the n-order derivatives of those functions. Therefore, the term AD must be reserved to conversion of code rather than to the "Optimized Taylor Method".

I strongly suggest to stay with these two terms above avoiding alternatives like "Power Series Method" or "Differential Transform Method", or "Picard method".

Acknowledgement 1. *This finalization of the Unifying view was made possible thanks to help of algebraists George Bergman and Alexander Givental*[3] *in the following way. In my search of possible equivalents of the Conjecture into the language of pure algebra, first I came to the Proposition 1 (in Appendix 1) missing the condition (2). When George Bergman brought Example 1 as a counterexample, I added that lacking condition. Then George Bergman circulated the so corrected Proposition*

[3] They both teach in the Berkeley university https://math.berkeley.edu/people/faculty

among his circle of professionals, of whom Alexander Givental had found the proof of it. I extend my cordial thanks to both.

Appendix 1: The crucial theorem by Alexander Givental[4]

The main obstacle in attempts of proving the Conjecture has been its claim of regularity of the target polynomial ODE $P(T,\ X_0, ...,\ X_n) = 0$, namely that

$$\left.\frac{\partial P}{\partial X_n}\right|_{t=t_0} \neq 0. \tag{17}$$

It was necessary to find an instrument, which makes such a claim.

In the theory of implicit equations we have the opposite situation when the regularity (17) is given, while existence and uniqueness of the solution follow from it. Here, therefore, we needed a kind of a vice versa statement formulated in the following Proposition.

Proposition 1. *Let $P(y,\ \mathbf{x})$ be an irreducible polynomial verifying that $y(\mathbf{x})$ is algebraic. As a full algebraic function, $y(\mathbf{x})$ may have several branches, say one of them being $y_i(\mathbf{x})$, passing through a point $(y_0,\ \mathbf{x}_0)$ whose small neighborhood is denoted U. Then,*

(1) *if $y_i(\mathbf{x})$ is holomorphic, and*
(2) *$y_i(\mathbf{x})$ is a unique solution of $P(y,\ \mathbf{x}) = 0$ in U,*

$$\left.\frac{\partial P}{\partial y}\right|_{(y_0,\ \mathbf{x}_0)} \neq 0.$$

A proof of this proposition was provided by Alexander Givental as the following Theorem.

Theorem 1. *Let P be a nonzero irreducible polynomial in variables $x_1, ...,\ x_n$ over the complex numbers, having constant term 0, and A a holomorphic function in $n-1$ complex variables defined in some neighborhood U of the origin, such that for some $\varepsilon > 0$, the zero-set of P on $U \times \{x_n|\ \ |x_n| < \varepsilon\}$ is the graph of $x_n = A(x_1,\ ...,\ x_{n-1})$. Then the coefficient in P of the monomial x_n is nonzero; i.e. $\dfrac{\partial P}{\partial x_n}$ is nonzero at the origin.*

Proof. We first observe that because P is irreducible, its gradient cannot be identically zero on its zero-set. Indeed, choose i such that $\dfrac{\partial P}{\partial x_i}$ is not identically zero on complex n-space. If $\dfrac{\partial P}{\partial x_i}$ were identically zero on the zero-set of P, then by the Hilbert Nullstellensatz, P would divide $\left(\dfrac{\partial P}{\partial x_i}\right)^m$ for some m, hence since the polynomial ring is a unique factorization domain and P is irreducible, P would divide $\dfrac{\partial P}{\partial x_i}$, which, looking at degrees in x_i, is impossible.

Hence the subset of the zero-set of P on which the gradient of P is zero must be a proper algebraic subset of that irreducible algebraic set, hence must have complex dimension $< n - 1$.

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[4]Alexander Givental provided the proof of this Theorem in private correspondence on December 8, 2022.

Now let H denote the ring of holomorphic functions on U. Dividing P by $x_n - A$ in $H[x_n]$, we get $P = (x_n - A)G + C$, with $C \in H$, $G \in H[x_n]$. Since P is identically zero on the set defined by $x_n = A$, we in fact have $C = 0$, so $P = (x_n - A)G$. In particular, the value of $\dfrac{\partial P}{\partial x_n}$ at the origin is the value of G at the origin, so it suffices the prove this nonzero. If G were zero at the origin, then since its zero-set is contained in that of P, and must have complex dimension at least $n - 1$, and $P = (x_n - A)G$ has zero gradient at all points of the zero-set of G, we would get a contradiction to the preceding paragraph. □

The following is a discussion of a few consequences (not necessary for the proof of the Conjecture.

Consider a version of Proposition 1 modified into a polynomial ODE.

Remark 11. *(The solution holomorphic and unique, yet the equation singular). Let $P(X_n;\ T,\ X_0, ...,\ X_{n-1})$ be an irreducible polynomial satisfied by a point $(t_0,\ a_0, ...,\ a_n)$, and $x(t)$, holomorphic at this point, is a unique solution of the ODE $P(x^{(n)};\ t,\ x, ...,\ x^{(n-1)}) = 0$. Unlike in the Proposition 1, the regularity (17) does not follow in this setting, as illustrated by the Examples 5, 6. A point $t = 0$ in those Examples is the point of violation of elementariness for both holomorphic functions, meaning that they can satisfy no regular polynomial ODE at this point. For ODEs, the regularity (17) does not follow from the fact that $x(t)$ is holomorphic and unique solution.*

Now consider the Proposition 1 for a scalar x.

Corollary 2. *(Ridding of self-crossing). If the condition (2) in Proposition 1 is violated because other branches $y_i(x)$ do pass through the same point $(y_0,\ x_0)$, there exists such an integer $k > 0$, that $y^{(k)}(x)$ is however a unique holomorphic branch at the small neighborhood U of the point $(y_0,\ x_0)$. Then, if $Q(Y_k,\ x)$ is an irreducible polynomial verifying that $y^{(k)}$ is algebraic so that $Q(y^{(k)},\ x) = 0$, all the conditions of the Proposition 1 are met for $y^{(k)}$ (rather than for y),*

$$\left.\frac{\partial Q}{\partial Y_k}\right|_{(y_0,\ x_0)} \neq 0,$$

and the ODE $Q(y^{(k)},\ x) = 0$ verifies that $y(x),\ y'(x),\ y''(x), ..,\ y^{(k-1)}(x)$ are elementary at x_0.

Remark 12. *(Difference between violation of elementariness and violation of the uniqueness of an algebraic solution). If the regularity (17) is violated for an ODE $P(x^{(n)};\ t,\ x, ...,\ x^{(n-1)}) = 0$ at a point t_0, application of the differentiating operator $\left(\dfrac{d}{dt}\right)^k$ to this ODE will not make it regular for whichever k, because the coefficient at the leading derivative would remain the same expression (17). On the contrary, if the regularity (17) is violated in the Proposition 1 because branches $y_i(x)$ pass through the same point $(y_0,\ x_0)$, differentiation of y into $y^{(k)}$ does lead to uniqueness of the branch passing through $(y_0,\ x_0)$ and fulfillment of regularity for the ODE $Q(y^{(k)},\ x) = 0$.*

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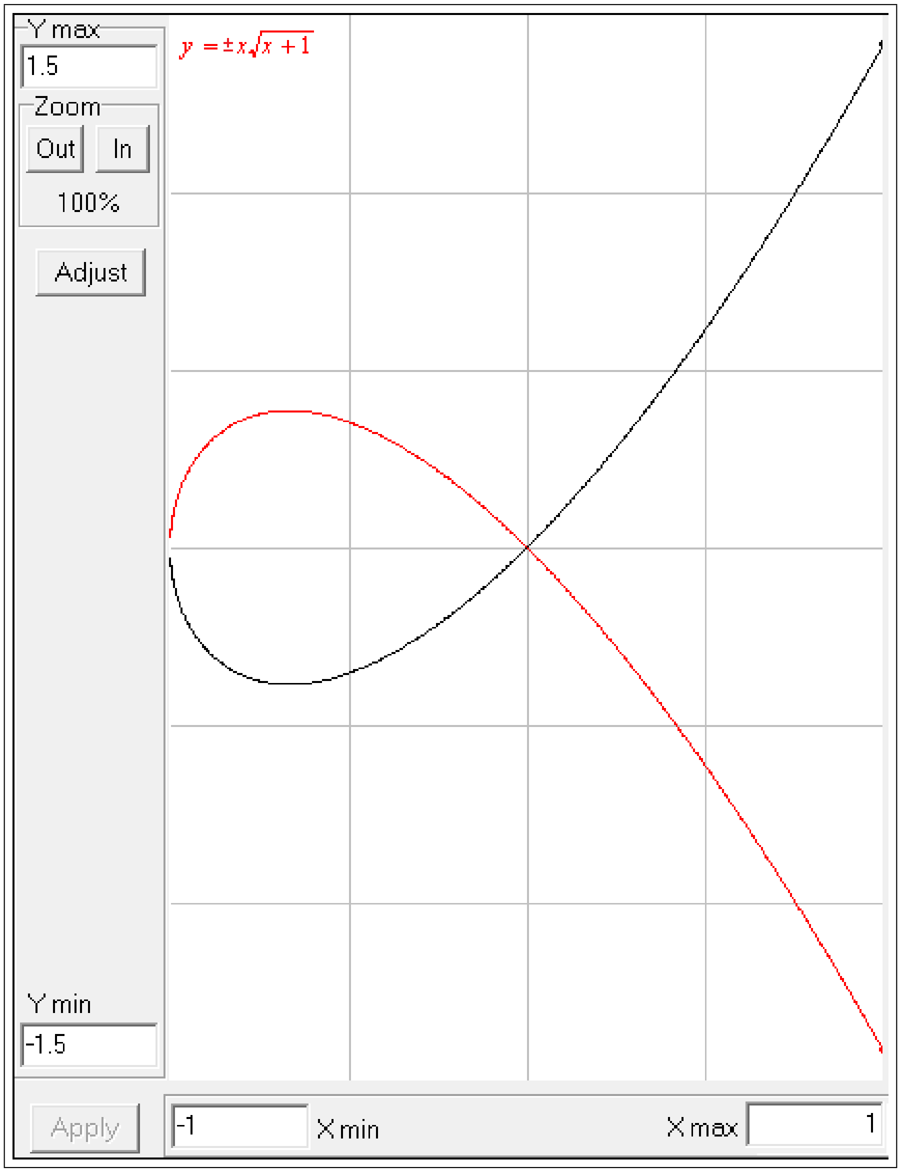

FIGURE 1. Graph of a full algebraic function $y = \pm x\sqrt{x+1}$ whose branches cross at the origin. The graph was generated via Taylor solver http://taylorcenter.org/Gofen/TaylorMethod.htm

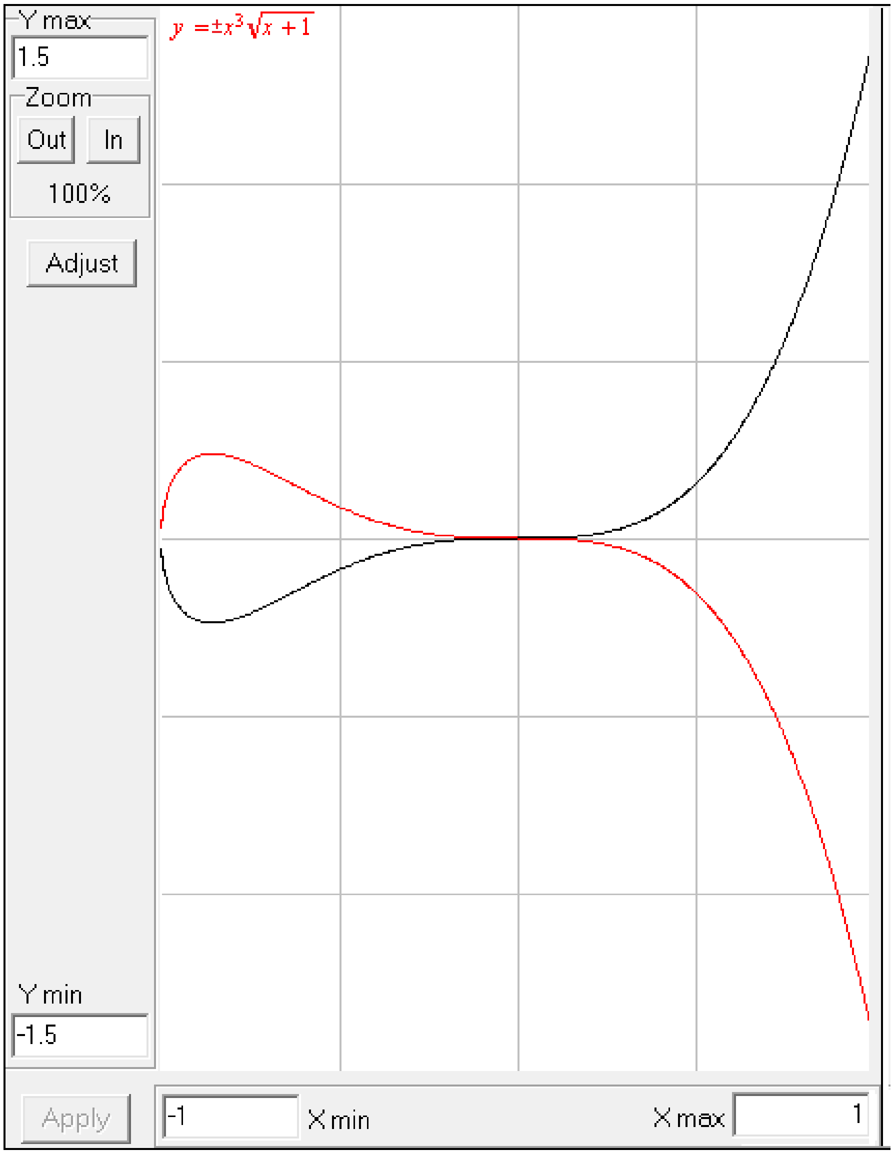

FIGURE 2. Graph of a full algebraic function $y = \pm x^3\sqrt{x+1}$ whose branches cross having tangency at the origin.