A stumbling problem

Alexander Gofen (April 2023)

This text explains what was an error and what remains a stumbling problem in an attempt to resolve the Conjecture and close the gap in the unifying view theory [1]. Though the manuscript in January 2023 was not accepted nor reviewed, yet it appeared as a preliminary draft (preprint) here [1]. The error is in Lemma 1, and the stumbling block now is the following problem.

Given an IVP for a polynomial system

$$\begin{aligned} x' &= P_1(t,\ x,\ y,\ z, ...), \quad & x|_{t=t_0} &= a, \\ y' &= Q_1(t,\ x,\ y,\ z, ...), \quad & y|_{t=t_0} &= b, \\ z' &= R_1(t,\ x,\ y,\ z, ...), \quad & z|_{t=t_0} &= c, \end{aligned} \tag{1}$$

we can obtain an infinite sequence of polynomial equations - the Fundamental Sequence

$x' = P_1(t,\ x,\ y,\ z, ...)$	$y' = Q_1(t,\ x,\ y,\ z, ...)$	$z' = ...$	...
... $x^{(k)} = P_k(t,\ x,\ y,\ z, ...)$ $x^{(k+1)} = P_{k+1}(t,\ x,\ y,\ z, ...)$ ...			

(2)

where the following recursive relations[1] take place:

$$\begin{aligned} P_{k+1}(t,\ x,\ y,\ z, ...) &= \frac{\partial P_k}{\partial t} + \frac{\partial P_k}{\partial x}x' + \frac{\partial P_k}{\partial y}y' + \frac{\partial P_k}{\partial z}z' ... \\ &= \frac{\partial P_k}{\partial t} + \frac{\partial P_k}{\partial x}P_1 + \frac{\partial P_k}{\partial y}Q_1 + \frac{\partial P_k}{\partial z}R_1 ... \end{aligned} \tag{3}$$

(and the similar infinite sequences may be written also for $y^{(k)},\ z^{(k)}, ...$ if we needed them).

Problem 1 *What to do if it happens that one of the columns in recurrences (3), say* $\left.\frac{\partial P_k}{\partial z}\right|_{t=t_0} = 0$ *for all* $k = 1,\ 2, ...$*? How to take advantage of this fact and to modify the source IVP (1) so that the sequence and the recurrences (2, 3) do not change, preserving the values* $x^{(k)}|_{t=t_0}$ *, yet to achieve one of the two goals: either* z *is eliminated, or at least one* $\left.\frac{\partial P_k}{\partial z}\right|_{t=t_0} \neq 0$.

[1] Simplicity of these recursive formulas hides an immense complexity of the final expressions $P_k(t,\ x,\ y,\ z, ...)$: more cumbersome than the multi-variate formula Faa-diBruno. That is because the Faa-diBruno formula still contains monomials like $(x^{(i)})^\alpha (x^{(j)})^\beta ... (y^{(i)})^\gamma ... (z^{(i)})^\delta ...$ with derivatives $y^{(i)},\ z^{(j)}$ rather than monomials over $x,\ y,\ z, ...$. The Faa-diBruno formula does not utilize the ODEs (1).

The fact that all partials $\left.\frac{\partial P_k}{\partial z}\right|_{t=t_0} = 0$ creates an illusion as though in such a special case it is possible to change R_1 arbitrarily due to its position in (3), yet this is not true. Changes in the polynomial R_1 would generally lead to changes in all polynomials P_k.

Examples

The case of linearly dependent columns (zero Jacobian)

Consider the IVP

$$\begin{aligned} x' &= y+z; \quad x(0)=a \\ y' &= y^2; \qquad y(0)=b \\ z' &= 2z^2; \qquad z(0)=c \end{aligned}$$

whose solution is

$$\begin{aligned} x &= ..., \\ y &= \frac{b}{1-bt} \text{ if } b\neq 0, \text{ or } y\equiv 0 \text{ if } b=0; \\ z &= \frac{c}{1-2ct} \text{ if } c\neq 0, \text{ or } z\equiv 0 \text{ if } c=0. \end{aligned}$$

The second and third ODEs are actually stand alone ODEs. We can write down their $n$derivatives

$$\begin{aligned} y^{(n)} &= n!y^{n+1} \\ z^{(n)} &= 2^n n! z^{n+1} \end{aligned}$$

and therefore we have expressions for F_n

$$x^{(n)} = P_n(x,\ y,\ z) = n!y^{n+1} + 2^n n! z^{n+1}$$

and

$$\frac{\partial P_n}{\partial y} = (n+1)!y^n; \qquad \frac{\partial P_n}{\partial z} = 2^n(n+1)!z^n.$$

The Jacobian J_{mn} of lines m and n, $n>m$ is

$$\begin{aligned} J_{mn} &= \begin{bmatrix} (n+1)!y^n & 2^n(n+1)!z^n \\ (m+1)!y^m & 2^m(m+1)!z^m \end{bmatrix} \\ &= 2^m(m+1)!z^m(n+1)!y^n - 2^n(n+1)!z^n(m+1)!y^m \\ &= 2^m(m+1)!(n+1)y^m z^m(y^{n-m} - (2z)^{n-m}). \end{aligned}$$

If the initial values are such that $b=2c$, all $J_{mn}|_{t=0} = 0$ meaning that columns $\left.\frac{\partial P_n}{\partial y}\right|_{t=0}$ and $\left.\frac{\partial P_n}{\partial z}\right|_{t=0}$ are linearly dependent when $y|_{t=0} = b = 2c$, $z|_{t=0} = c$, namely

$$\left.\frac{\partial P_n}{\partial z}\right|_{t=0} = 2^n(n+1)!c^n; \quad \left.\frac{\partial P_n}{\partial y}\right|_{t=0} = (n+1)!(2c)^n$$

$$\left.\frac{\partial P_n}{\partial z}\right|_{t=0} = \left.\frac{\partial P_n}{\partial y}\right|_{t=0}.$$

Now observe, that with such special initial values $b = 2c$ we can see that y and z are related:

$$y = \frac{b}{1-bt} = \frac{2c}{1-2ct}$$

$$z = \frac{c}{1-2ct}$$

i.e. $y \equiv 2z$, being an integral of this IVP for these special initial values.

The case of a zero column

Consider the same IVP when $b = 0, \ c \neq 0$. Now we see that $\left.\frac{\partial P_n}{\partial y}\right|_{t=0} = 0$ for all n. Observe again, that with these special initial values, the solution component $y = const = 0$.

Summary 2 *We do not know, however, what follows from the linear dependence of the columns, or from one column being zero in the fundamental sequence for* general *IVPs.*

1. The Gap in the Unifying View Closed. (Actually, not yet).
   https://academia.edu/98194003/The Gap in the Unifying View Closed
   https://researchsquare.com/article/rs-2494232/v1

Appendix

Just some observations. Viewing $\{x^{(k)}\}$, $\{y^{(k)}\}$, $\{z^{(k)}\}$ as infinite vectors (columns) and writing down the fundamental sequence for each of them, say for $x^{(k)}$ (formulas (3)), we see that

$$\{x^{(k)}\} = \begin{pmatrix} \dots & \dots & \dots & \dots \\ \frac{\partial P_k}{\partial t} & \frac{\partial P_k}{\partial x} & \frac{\partial P_k}{\partial y} & \frac{\partial P_k}{\partial z} \\ \frac{\partial P_{k+1}}{\partial t} & \frac{\partial P_{k+1}}{\partial x} & \frac{\partial P_{k+1}}{\partial y} & \frac{\partial P_{k+1}}{\partial z} \\ \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} 1 \\ x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} 1 \\ x' \\ y' \\ z' \end{pmatrix}$$

What is the "meaning" of the columns

$$\begin{pmatrix} \dots \\ \frac{\partial P_k}{\partial x} \\ \frac{\partial P_{k+1}}{\partial x} \\ \dots \end{pmatrix}, \quad \begin{pmatrix} \dots \\ \frac{\partial P_k}{\partial y} \\ \frac{\partial P_{k+1}}{\partial y} \\ \dots \end{pmatrix}, \dots$$

of the infinite matrix A, whose elements are polynomials? What if rank of this matrix $A|_{t=t_0}$ at a point happens to be less than the maximal possible (in this case 4), meaning that at this point its numerical columns are linearly dependent (or one of them is zero)?

Experiment.

Does the condition $\left.\frac{\partial P_1}{\partial z}\right|_{t=t_0} = 0$ propagate further down?

$k = 1$. $P_1(t,\ x,\ y,\ z)$ and and we assume that $\left.\frac{\partial P_1}{\partial z}\right|_{t=t_0} = 0.$

$k = 2$.

$$\begin{aligned} P_2 &= \frac{\partial P_1}{\partial t} + \frac{\partial P_1}{\partial x} P_1 + \frac{\partial P_1}{\partial y} Q_1 + \frac{\partial P_1}{\partial z} R_1. \\ \frac{\partial P_2}{\partial z} &= \frac{\partial^2 P_1}{\partial t \partial z} + \frac{\partial P_1}{\partial x}\frac{\partial P_1}{\partial z} + \frac{\partial P_1}{\partial y}\frac{\partial Q_1}{\partial z} + \frac{\partial P_1}{\partial z}\frac{\partial R_1}{\partial z} + \\ &\quad + \frac{\partial^2 P_1}{\partial x \partial z} P_1 + \frac{\partial^2 P_1}{\partial y \partial z} Q_1 + \frac{\partial^2 P_1}{\partial z \partial z} R_1. \end{aligned}$$

At $t = t_0$

$$\left.\frac{\partial P_2}{\partial z}\right|_{t=t_0} = \left(\begin{array}{c} \frac{\partial^2 P_1}{\partial t \partial z} + \frac{\partial P_1}{\partial y}\frac{\partial Q_1}{\partial z} + \\ + \frac{\partial^2 P_1}{\partial x \partial z} P_1 + \frac{\partial^2 P_1}{\partial y \partial z} Q_1 + \frac{\partial^2 P_1}{\partial z \partial z} R_1 \end{array} \right)_{t=t_0}.$$

What does it take that this expression be zero at $t = t_0$?