Unremovable 'removable' singularities

Alexander GOFEN

Dedicated to the memory of Prof. Michael Lidov

Abstract. The article attempts to answer the question why the so called 'removable' or 'regular' singularities in certain holomotphic functions cannot be 'removed'. This problem may be understood in the frame of the generalized *elementary functions* (i.e. functions defined as solutions of explicit rational Ordinary Differential Equations). Along with several known examples, the article produces a family of infinitely many functions having 'regular singularities'. There are formulated also two open questions.

Introduction

The concept of *removable* (or *regular*) singularities emerges when an holomorphic function $x(t)$ is presented either as a formula, or as a solution of an Initial Value Problem (IVP) for ODEs, invalid at an isolated point, say $t = 0$, yet valid in its neighborhood. If by convention the proper value is assigned to $x(t)$ at $t = 0$, the function at this point becomes holomorphic, so that its 'seeming singularity' is 'removed'. That is, the singularity contained in the formula or the equations defining the function, does not necessarily belong to this function: for example, this branch of an algebraic function

$$x(t) = \frac{+\sqrt{1+t} - 1}{t}, \qquad x|_{t=0} = \frac{1}{2}$$

is holomorphic at $t = 0$ whose algebraicity is demonstrated by the polynomial equation $tx^2 + 2x - 1 = 0$. However, this branch of the function $x(t)$ may be expressed also regularly

$$x(t) = \frac{1}{+\sqrt{1+t} + 1},$$

and as the solution of a regular at $t = 0$ IVP

$$x' = -\frac{x^2}{2tx + 2}, \qquad x|_{t=0} = \frac{1}{2}.$$

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This paper owes a lot to (hot) discussions with Harley Flanders. The beautiful example $x(t) = \cos\sqrt{t}$ is by courtesy of George Bergman.

An entire function $x = te^t$ is represented via a regular formula, yet ODEs defining it may be either singular or regular at $t = 0$ (Item 10, Table 1).

However there exist functions for which all currently known formulas or ODEs have a singularity at an isolated point, even though the functions themselves are holomorphic at this point. The examples of such functions are $x(t) = \dfrac{e^t - 1}{t}$, $x(t) = \cos\sqrt{t}$, the solution of the IVP

$$tx'' - x = 0; \qquad x|_{t=0} = 0; \quad x'|_{t=0} = 1,$$

and each of functions 1-7, Table 1.

Is it possible that regular ODEs representing these functions exist, but are not yet known? This question makes sense only if we specify in which class of equations we are looking for the answer. If the right hand sides of the ODEs are allowed to be any holomorphic functions (obtainable or not), the answer is trivial and tautological: introduce an ODE $x' = \varphi(t)$, where a holomorphic $\varphi(t) = x'(t)$.

We, however, consider a subclass of holomorphic right-hand sides called generalized *elementary* functions (first introduced by R. Moore [1]). This class widens the conventionally defined (by Liouville) elementary functions to include practically all functions used in applications. In simplest terms, generalized *elementary* functions [2] are those which may be defined as solutions of IVPs for explicit ODEs having rational right hand sides regular at the initial point. The goal is to prove that $x(t) = \dfrac{e^t - 1}{t}$ and several other functions (Items 1-7, Table 1) cannot satisfy any rational regular ODE at $t = 0$.

All throughout this paper functions and solutions of ODEs are considered as holomorphic functions in complex space $\mathbf{C}$.

Polynomial ODEs having the same solution

If we are given a polynomial ODE having a solution $x(t)$, it is possible to obtain a non-trivial family of polynomial ODEs (not necessarily just multiplied by a non-zero factor), still having the same solution $x(t)$. We are particularly interested in the case when all derivatives of $x(t)$ are rational at $t = 0$.

LEMMA 1. *Let an holomorphic function $x(t)$ at the neighborhood of $t = 0$ satisfy a nontrivial polynomial ODE*

$$F(t,\ x,\ x',\ \cdots,\ x^{(m)}) = \sum_{k=1}^{q} a_k t^\alpha x^\beta (x')^\gamma ... (x^{(m)})^\omega = 0, \tag{1}$$

$$x^{(m)}\Big|_{t=0} = r_m, \quad m = 0, 1, 2, ... \tag{2}$$

with complex *coefficients a_k (k is omitted at power indexes $\alpha_k,\ \beta_k, ...$). Then this $x(t)$ also satisfies infinitely many polynomial ODEs, such that their coefficients are solutions of a special linear algebraic system 4.*

In particular, if all derivatives $x^{(m)}\big|_{t=0}$ are rational, there exists a polynomial ODE with rational *coefficients satisfied by $x(t)$.*

PROOF. Obtain a sequence of polynomial ODEs by differentiating 1:

$$F_0(t,\ x,\dots,\ x^{(m)}) = F = \sum a_k t^\alpha x^\beta (x')^\gamma ...(x^{(m)})^\omega = \sum a_k M_{0k} = 0;$$
$$\frac{d}{dt}F_0(t,\ x,\ \dots,\ x^{(m)}) = F_1(t,\ x,\dots,\ x^{(m)},\ x^{(m+1)}) = \sum a_k \frac{d}{dt} M_{0k} =$$
$$= \sum a_k (\alpha t^{\alpha-1} x^\beta (x')^\gamma ...(x^{(m)})^\omega + \beta t^\alpha x^{\beta-1} (x')^{\gamma+1} ...(x^{(m)})^\omega + ...$$
$$+\ \omega t^\alpha x^\beta (x')^\gamma ...(x^{(m)})^{\omega-1} x^{(m+1)}) = \sum a_k M_{1k} = 0;$$

. . . .

(3)
$$\frac{d^n}{dt^n}F_0(t,\ x,\dots,\ x^{(m)}) = F_n(t,\ x,\dots,\ x^{(m)},...,\ x^{(m+n)}) = \sum_{k=1}^{q} a_k M_{nk} = 0$$

. . . .

Here values M_{0k} denote monomials of polynomial 1, while M_{nk} are n-order derivatives of those monomials. Substitute the initial values 2 into equations 3, obtaining a linear algebraic system

(4) $$a_1 M_{n1} + a_2 M_{n2} + ... + a_q M_{nq} = 0, \qquad n = 0, 1, 2, ...$$

in $a_1, a_2, ...a_q$. System 4 is an infinite over-defined yet solvable linear homogeneous system, having a non-zero solution — coefficients of the given polynomial 1 by the condition of the Lemma. Therefore system 4 has infinitely many solutions (say $a_1, a_2, ...a_q$ multiplied by a factor, and possibly others).

Let $b_1, b_2, ...b_q$ be any of those solutions. It generates a polynomial equation

(5) $$G_0 = \sum b_k T^\alpha Y_0^\beta Y_1^\gamma ... Y_m^\omega = 0$$

differing from the given 1

$$F_0 = \sum a_k T^\alpha X_0^\beta X_1^\gamma ... X_m^\omega = 0$$

in the coefficients at the corresponding monomials.

We are going to prove that $x(t)$ satisfies any polynomial G_0 with coefficients $b_1, b_2, ...b_q$ obtained as a solution of the linear system 4. Substitute $x(t)$ into G_0, denoting a non-zero deviation as $\varepsilon(t)$:

$$G_0(t,\ x,\dots,\ x^{(m)}) = \sum_{k=1}^{q} b_k t^\alpha x^\beta (x')^\gamma ...(x^{(m)})^\omega = \varepsilon(t).$$

Apply differentiation to G_0, obtain an infinite system analogous to 3, and observe that at $t = 0$

$$G_n(t,\ x,\dots,\ x^{(m)},...,\ x^{(m+n)})|_{t=0} = \varepsilon^{(n)}(t)|_{t=0} = 0, \qquad n = 0, 1, 2, ...$$

for all n. Therefore, as an holomorphic function, $\varepsilon(t) \equiv 0$, so that $x(t)$ does satisfy any polynomial ODE generated by the linear system 4. This proves the first statement of the Lemma.

Now assume that all values r_m of derivatives 2 are rational. In order to obtain the general solutions of 4, consider the matrix

$$M = \|M_{ij}\| \qquad 0 \leqslant i < \infty, \quad 1 \leqslant j \leqslant q$$

of the system. This matrix (and the linear system) is infinite only in the number of rows (equations). Only finite number of them are linearly independent. Let the maximal number of linearly independent equations 4 be $p > 0$, $p < q$. Therefore there must exist p independent variables with a nonzero sub-determinant corresponding to them, and $q - p$ dependent variables. Among $b_1, b_2, ...b_q$, consider p those which are independent, and assign them rational values. Then the remaining dependent variables must all be rational too (as ratios of sub-determinants of matrix M, whose all elements are rational numbers). The obtained rational coefficients $b_1, b_2, ...b_q$ generate the polynomial G_0 having the solution $x(t)$, which completes the proof. □

EXAMPLE 1. *As an illustration, consider an holomorphic element* $x^{(m)}\big|_{t=0} = m!$, $m = 0, 1, 2, ...$*(representing* $x = \frac{1}{1-t}$ *indeed), and an implicit polynomial equation*

$$Ax^2 + Bxt + Cx't + Dx + Ex' + F = 0 \tag{6}$$

whose coefficients $A, B, ...$ *are to be determined. By differentiation and substitution of the initial values obtain*

$$\begin{array}{l} A \qquad\qquad\qquad\qquad\qquad + D \quad + E \qquad + F = 0 \\ A(m+1)! + Bm! + Cmm! + Dm! + E(m+1)! = 0, \quad m = 1, 2, ... \end{array}$$

The general solution of this system

$$B = -A - D - E, \quad C = -A - E, \quad F = -A - D - E$$

delivers infinitely many solutions. In particular, the three solutions below exemplify different polynomial equations all satisfied by $x(t)$*:*

$E = 0,\ A = 0,\ D = 1$ $B = -1,\ C = 0,\ F = -1$	$E = 1,\ A = -1,\ D = 0$ $B = 0,\ C = 0,\ F = -1$	$E = 1,\ A = 0,\ D = -1$ $B = 0,\ C = -1,\ F = 0$
$x - xt - 1 = 0$	$x' - x^2 = 0$	$x' - x't - x = 0$

No regular representation for $\frac{e^t-1}{t}$

We deal with the entire function

$$x(t) = \frac{e^t - 1}{t}\ , \quad x|_{t=0} = 1.$$

It is easily checked that

$$x^{(m)}\Big|_{t=0} = \frac{1}{m+1}, \quad m = 0, 1, 2, ... \tag{7}$$

THEOREM 1. *The function* $x(t)$ *cannot be a solution of any non-trivial, implicit, polynomial ODE*

$$F(t,\ x,\ x',\ \cdots,\ x^{(m)}) = 0$$

with integer coefficients in the corresponding polynomial

$$F(T,\ X_0,\ X_1,\ \cdots,\ X_m),$$

having

$$\left.\frac{\partial F}{\partial X_m}\right|_{t=0} \neq 0.$$

PROOF. Denote

$$F_0(t,\ x,\ldots,\ x^{(m)}) = F = \sum a_k t^\alpha x^\beta (x')^\gamma ... (x^{(m)})^\omega = 0 \tag{8}$$

where a_k are integers (k is omitted at power indexes $\alpha_k,\ \beta_k, ...$).

Repeatedly differentiate relation 8, denoting the result of N differentiations by

$$F_N(t,\ x,\ldots,\ x^{(m)},...,\ x^{(m+N)}) = \frac{d^N}{dt^N} F_0(t,\ x,\ldots,\ x^{(m)}).$$

Prove by the induction, that in each of polynomials F_N the highest derivative $x^{(m+N)}$ appears only in one expression always with the same factor $\frac{\partial F_0}{\partial X_m}$. Observe, that

$$F_1 = \frac{d}{dt} F_0(t,\ x,\ldots,\ x^{(m)}) = \frac{\partial F_0}{\partial X_m} x^{(m+1)} + Q_0(t,\ x,\ldots,\ x^{(m)});$$

$$F_2 = \frac{d}{dt} F_1(t,\ x,\ldots,\ x^{(m+1)}) = \frac{\partial F_1}{\partial X_{m+1}} x^{(m+2)} + Q_1(t,\ x,\ldots,\ x^{(m+1)}) =$$

$$= \frac{\partial F_0}{\partial X_m} x^{(m+2)} + Q_1(t,\ x,\ldots,\ x^{(m+1)}).$$

Assuming

$$F_N = \frac{\partial F_0(t,\ x,\ldots,\ x^{(m)})}{\partial X_m} x^{(m+N)} + Q_{N-1}(t,\ x,\ldots,\ x^{(m+N-1)});$$

$$\frac{\partial F_{N-1}(t,\ x,\ldots,\ x^{(m+N-1)})}{\partial X_{m+N-1}} = \frac{\partial F_0}{\partial X_m}$$

to be true for N, obtain

$$F_{N+1} = \frac{d}{dt} F_N(t,\ x,\ldots,\ x^{(m+N)}) = \frac{\partial F_N}{\partial X_{m+N}} x^{(m+N+1)} + Q_N(t,\ x,\ldots,\ x^{(m+N)}) =$$

$$= \frac{\partial}{\partial X_{m+N}} \left(\frac{\partial F_0}{\partial X_m} \underbrace{x^{(m+N)}}_{\text{the only occurrence of } x^{(m+N)} \text{ in } F_N} + Q_{N-1}(t,\ x,\ldots,\ x^{(m+N-1)}) \right) x^{(m+N+1)} +$$

$$+ Q_N(t,\ x,\ldots,\ x^{(m+N)}) = \frac{\partial F_0}{\partial X_m} x^{(m+N+1)} + Q_N(t,\ x,\ldots,\ x^{(m+N)}).$$

Observe that the polynomials F_N have integer coefficients. By the condition of this Theorem, $\left.\frac{\partial F_0}{\partial X_m}\right|_{t=0} = A \neq 0$. As $x^{(k)}|_{t=0} = \frac{1}{k+1}$, the value A is rational. Multiply F_0 by a proper integer to clear all denominators so that value $\left.\frac{\partial F_0}{\partial X_m}\right|_{t=0} = A$ becomes an integer. Then the equation for F_N takes the form:

$$F_N = \frac{A}{m+N+1} + Q_{N-1}(t, x, \ldots, x^{(m+N-1)}) = 0 \tag{9}$$

With growing N, the denominator $m+N+1$ will become greater than A, and then it will reach some prime $p = m+N+1$ so that $\frac{A}{p}$ is a fraction in the lowest terms. All the remaining terms in $Q_{N-1}(t, x, \ldots, x^{(m+N-1)})$ must be integers or fractions, whose denominators contain primes less than p. Thus the isolated fraction $\frac{A}{p}$ and $Q_{N-1}(t, x, \ldots, x^{(m+N-1)})$ cannot cancel, which is impossible, proving the Theorem. □

Unlike the previous, the next theorem deals with ODEs having complex (non-integer) coefficients.

THEOREM 2. *The function $x(t)$ cannot be a solution of any non-trivial, implicit, polynomial ODE with complex coefficients*

$$F(t,\ x,\ x',...,\ x^{(m)}) = 0 \tag{10}$$

having

$$\left.\frac{\partial F}{\partial X_m}\right|_{t=0} \neq 0. \tag{11}$$

PROOF. Assume the opposite, that ODE 10 has $x(t)$ as a solution in a neighborhood of $t = 0$.

Step 0: From complex to real coefficients. Observe, that $x(t)$ and all its derivatives satisfying polynomial 10, are real-valued functions on the real axis. Assume therefore the coefficients of polynomial 10 are real.

Step 1: From irrational to rational coefficients. According to Lemma 1, $x(t)$ must satisfy infinitely many nontrivial polynomial ODEs with rational coefficients $b_1, b_2, ...b_q$, obtainable as solutions of the linear algebraic equation 4. The coefficients $a_1, a_2, ...a_q$ of 10 are the solutions of linear system 4 too. Among them consider the independent ones a_k, and choose their rational approximation b_k so close to a_k, that for the modified polynomial G_0 (Lemma 1, equation 5) corresponding to the complete set of rational coefficients $b_1, b_2, ...b_q$, condition 11 still holds. To not complicate notation, assume that the given equation 10 already has all rational coefficients.

Step 3: Apply a proper integer factor to the polynomial equation 10 (having rational coefficients) to clear all denominators. Now $x(t)$ satisfies a polynomial equation with integer coefficients — impossible, according to Theorem 1, which proves this theorem. □

COROLLARY 1. *The function $x(t)$ cannot be a solution of an IVP for any explicit rational ODE*

$$x^{(m+1)} = \frac{P(t,\ x,\ x',...,\ x^{(m)})}{Q(t,\ x,\ x',...,\ x^{(m)})} \tag{12}$$

having the denominator

$$Q|_{t=0} \neq 0, \tag{13}$$

nor indeed it can be a solution of an IVP for any explicit polynomial ODE

$$x^{(m+1)} = P(t,\ x,\ x',...,\ x^{(m)}).$$

The proof of this corollary relies on the following

LEMMA 2. *The implicit polynomial ODE 10 non-singular at $t = 0$ (Condition 11) and the explicit rational ODE 12 with a nonzero denominator (Condition 13) converts into each other.*

PROOF. Really, in a rational ODE 12 written as a polynomial equation

$$F = x^{(m+1)}Q(t,\ x,\ x',...,\ x^{(m)}) - P(t,\ x,\ x',...,\ x^{(m)}) = 0$$

derivative $\left.\frac{\partial F}{\partial X_{n+1}}\right|_{t=0} = Q|_{t=0} \neq 0$. Inversely, if a polynomial ODE 10 is given, apply $\frac{d}{dt}$

$$\frac{\partial F}{\partial T} + \frac{\partial F}{\partial X}x' + ... + \frac{\partial F}{\partial X_{m-1}}x^{(m)} + \frac{\partial F}{\partial X_m}x^{(m+1)} = 0$$

and obtain a rational ODE relying on condition 11

$$x^{(m+1)} = -\frac{\frac{\partial F}{\partial T} + \frac{\partial F}{\partial X}x' + ... + \frac{\partial F}{\partial X_{m-1}}x^{(m)}}{\frac{\partial F}{\partial X_m}}$$

□

Proof. (The Corollary). Assume that the rational ODE 12 exists under condition 13. According to the Lemma, rational ODE 12 converts to the polynomial one. That is impossible according to Theorem 2, which proves this corollary. □

Other functions having no regular representation

The method of proof in Theorem 1 applies not only to $x(t)$ having expansion 7, but also to infinitely many other holomorphic functions defined by a variety of expansions (Examples 2-7, Table 1).

Corollary 2. *Let* $H(n) \neq 0$ *be an integer-valued function such that the maximal prime* $p \leqslant n$ *occurs among the factors of* $H(n)$*, and let* $G(n)$ *be an integer-valued function, whose factors do not exceed* n*. Then the statement of Theorem 1 holds also for functions defined by an analytic element*

$$x^{(n)}|_{t=0} = \begin{cases} \dfrac{G(n-1)}{H(n)} & \textit{for infinitely many prime values of } n \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Reconsider equation 9 in Theorem 1, which takes the form

$$F_N = \frac{AG(m+N-1)}{H(m+N)} + Q_{N-1}(t,\ x, \dots,\ x^{(m+N-1)}) = 0. \tag{14}$$

Choose such a big $n = m + N$, that n is prime, $n > A$. Then $\frac{AG(m+N-1)}{H(m+N)}$ is a fraction in the lowest terms, cancellation in equation 14 is impossible, proving this corollary. □

It is easy to see that Examples 2-7, Table 1, meet the condition of Corollary 2. Two more examples (not in the Table), defined by their expansion at one point only, also do: $x^{(n)}|_{t=0} = \frac{1}{n!}$, and $x^{(n)}|_{t=0} = \frac{1}{n^n}$. (Other representations of these entire analytic functions are not known).

Discussion

Another proof of Theorem 2 belongs to H. Flanders [3,4]. Moreover, he proved that among ODEs of *first* order defining $x = \frac{e^t-1}{t}$, the known ODEs

$$x' = R(x) = \frac{tx - x + 1}{t}$$

$$P(t,\ x,\ x') = tx' - tx + x - 1 = 0 \tag{15}$$

are unique in the sense, that any implicit first order polynomial ODE divides by P, while any explicit rational first order ODE reduces to R.

Table 1 summarizes the functions considered in the article. Items (1-7) have no regular representation. Formulas for functions (8,9) are regular at $t = 0$: they are entered into the Table for comparison only. (There exist both regular and singular ODEs for function (8) and (12). We do not know any non-singular rational ODE for the Bessel functions (11), nor is Corollary 2 applicable to them.

Taylor expansions for elementary functions. Although Theorem 2 and Corollaries 1, 2 for functions (1-7) in Table 1 are about certain specialty of the point $t = 0$ in these functions, it is not yet known whether these functions are *non-elementary* at this isolated point. In order to prove it, a stronger theorem should be established (see the *Proposition* in the next section). We can only suspect that $x(t)$ is possibly non-elementary at $t = 0$. If so, then any system of rational ODEs satisfied by $x(t)$ must be singular, so that specialty of the point $t = 0$ in $x(t)$ is 'unremovable' in the class of elementary functions.

	Functions	ODEs	Derivatives at $t = 0$
1	$x = \frac{e^t-1}{t}$	$x' = \frac{tx-x+1}{t}$	$x^{(n)} = \frac{1}{n+1}$
2	$x = \frac{\sin t}{t}$ $y = \cos t$ $z = \sin t$	$x' = \frac{y-x}{t}$ $y' = -z$ $z' = y$	$x^{(n)} = \frac{(-1)^{n/2}}{n+1}$ even n, or 0 $y^{(n)} = (-1)^{n/2}$ even n, or 0 $z^{(n)} = (-1)^{(n+1)/2}$ odd n, or 0
3	$x = \frac{\cos t-1}{t^2}$ $y = \cos t$ $z = \sin t$	$x' = \frac{2-2y-tz}{t^3}$ $y' = -z$ $z' = y$	$x^{(n)} = \frac{(-1)^{n/2+1}}{(n+1)(n+2)}$ even n, or 0 $y^{(n)} = (-1)^{n/2}$ even n, or 0 $z^{(n)} = (-1)^{(n+1)/2}$ odd n, or 0
4	$x = \cos\sqrt{t}$ $y = \sin\sqrt{t}$ $z = \sqrt{t}$	$x' = -\frac{yz}{2t}$ or $x'' = -\frac{x+2x'}{4t}$ $y' = \frac{xz}{2t}$ $z' = \frac{z}{2t}$	$x^{(n)} = (-1)^n \frac{n!}{(2n)!}$ singular singular
5	$x = \frac{\cos\sqrt{t}-1}{t}$ $u = \cos\sqrt{t}$ $v = \sin\sqrt{t}$ $z = \sqrt{t}$	$x' = \frac{-vz-2u+2}{2t^2}$ $u' = -\frac{vz}{2t}$ $v' = \frac{uz}{2t}$ $z' = \frac{z}{2t}$	$x^{(n)} = (-1)^{n+1} \frac{(n-1)!}{(2n)!}$ $u^{(n)} = (-1)^n \frac{n!}{(2n)!}$ singular singular
6	$tx'' - x = 0$	$x'' = \frac{x}{t}$	$x^{(n)} = \frac{1}{(n-1)!},\ n \geqslant 1,\ x(0) = 0$
7	$x = \frac{\ln(t+1)}{t}$	$x' = \frac{1-tx-x}{t(t+1)}$	$x^{(n)} = \frac{(-1)^{n+1}n!}{n+1}$
8	$x = \ln(t+1)$	$x' = \frac{1}{t+1}$	$x^{(n)} = (-1)^{n-1}(n-1)!,\ x(0) = 0$
9	$x = e^t$	$x' = x$	$x^{(n)} = 1$
10	$x = te^t$	$x' = \frac{x}{t} + x$ or $x'' = 2x' - x$	$x^{(n)} = n$
11	Bessel functions $J_p,\ p = 0,\ 1,\ 2, ...$	$x'' = \frac{-tx'-(t^2-p^2)x}{t^2}$	$x^{(n)} = \frac{(-1)^k C^k_{2k+p}}{2^{2k+p}},\ \ n = 2k + p$ or 0
12	Lambert function	$x(t)e^{x(t)} = t$; $x' = \frac{x}{t(x+1)}$ or $x'' = (x')^2(x't - 2)$	$x^{(n)} = (-1)^{n-1} n^{n-1},\ x(0) = 0$

Table 1. Summary of functions, ODEs defining them, and their n-order derivatives

Elementary functions represent practically all functions used in applications, and they are elementary (almost) at all points of their holomorphy. Yet their Taylor expansions have certain specialty, distinguishing them from non-elementary functions: their Taylor coefficients are obtainable via a fixed number of *explicit* formulas of Automatic Differentiation (AD), corresponding to a system of *explicit* rational ODEs and algebraic relations [2]. Systems of *implicit* rational ODEs and *implicit* algebraic relations are considered by Nedialkov and Pryce [5]. Generally, an expansion generated by an arbitrary recursive formula or algorithm may not be expected to represent a function being elementary at this or other points.

Open statements. The method of proof of Theorem 2 for an n-order ODE is not applicable to *systems* of ODEs, leaving open the following

PROPOSITION 1. *An entire function*

$$x(t) = \frac{e^t - 1}{t}, \qquad x^{(m)}|_{t=0} = \frac{1}{m+1},\ m = 0, 1, 2, ...$$

at the point $t = 0$ *cannot be a solution of an IVP for any system of rational ODEs*

$$\begin{aligned} x' &= \frac{P_1(t,\ x,\ y,\ z, ...)}{Q_1(t,\ x,\ y,\ z, ...)} \\ y' &= \frac{P_2(t,\ x,\ y,\ z, ...)}{Q_2(t,\ x,\ y,\ z, ...)} \\ &............ \end{aligned} \tag{16}$$

whose all denominators $Q_i|_{t=0} \neq 0$*, nor indeed it can be a solution of an IVP for any system of explicit polynomial ODEs*

$$\begin{aligned} x' &= P_1(t,\ x,\ y,\ z, ...) \\ y' &= P_2(t,\ x,\ y,\ z, ...) \\ &............ \end{aligned}$$

If proved, this Proposition would establish existence of a new type of special points in elementary holomorphic functions (along with Poles, Branching, and Essential singularities).

Another open statement (which, if proved, would solve Proposition 1), is the following

CONJECTURE 1. *Consider an IVP for a system of rational ODEs 16 with nonzero denominators at a given point* $(t_0,\ x_0,\ y_0,\ z_0, ...)$ *of the phase space so that the IVP has a unique holomorphic solution* $(x(t),\ y(t),\ z(t), ...)$ *in a neighborhood of* t_0*. In particular, all derivatives* $x^{(k)}|_{t=t_0} = a_k,\ \ k = 0, 1, 2, ...$ *Then there exists an explicit rational ODE of order* $n \geqslant 1$

$$x^{(n)} = \frac{F(t,\ x, ...,\ x^{(n-1)})}{G(t,\ x, ...,\ x^{(n-1)})}; \qquad x^{(k)}|_{t=t_0} = a_k, \quad k = 0, 2, ...n - 1$$

whose denominator $G(t_0,\ a_0, ...,\ a_{n-1}) \neq 0$*, so that the IVP at* $(t_0,\ a_0, ...,\ a_{n-1})$ *has* $x(t)$ *as a unique holomorphic solution. Or there exists an implicit polynomial ODE*

$$H(t,\ x, ...,\ x^{(n-1)},\ x^{(n)}) = 0$$

regular at the point $(t_0,\ a_0, ...,\ a_n)$*, i.e.* $\dfrac{\partial H(t_0,\ a_0, ...,\ a_n)}{\partial X_n} \neq 0$*,* $\ (X_n = x^{(n)})$*, so that the IVP at* $(t_0,\ a_0, ...,\ a_n)$*,* $H(t_0,\ a_0, ...,\ a_n) = 0$*, has* $x(t)$ *as a unique holomorphic solution.*

Remark 1. *Here polynomial H may be assumed linear in $x^{(n)}$. If it isn't, differentiate it so that*

$$\frac{dH}{dt} = \frac{\partial H(t,\ x, ...,\ x^{(n)})}{\partial X_n} x^{(n+1)} + \frac{\partial H(t,\ x, ...,\ x^{(n)})}{\partial X_{n-1}} x^{(n)} + ...$$

is already linear in the now leading derivative $x^{(n+1)}$. Regularity of this ODE depends on the same factor $\dfrac{\partial H(t_0,\ a_0, ...,\ a_n)}{\partial X_n}$.

The Conjecture claims convertibility of an explicit first order system of rational ODEs regular at a point into one explicit rational ODE of order n regular at this point. (The opposite conversion from one n-order ODE into a system of first order ODEs is well known and trivial).

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Appendix of 2024

Jim Sochacki suggested a way of generation infinitely many functions having the points - suspects of losing elementariness similar to the fractional functions in Table 1.

Let a function $f(t)$ be holomorphic in a vicinity of t_0. Consider the function

$$g(t) = \begin{cases} \dfrac{f(t) - f(t_0)}{t - t_0}, & t \neq t_0 \\ f'(t_0) \end{cases} . \tag{17}$$

Theorem 3. *If $f(t)$ is holomorphic in a vicinity of t_0, so is $g(t)$ in the same vicinity.*

Proof. The function $g(t)$ is obviously holomorphic for $t \neq t_0$. At $t = t_0$ the derivative $f'(t_0)$ exists because $g(t)$ is holomorphic. Observe, that function $g(t)$ so defined at $t = t_0$ is continuous at $t = t_0$ because $\lim_{t \to t_0} g(t) = f'(t_0) = g(t_0)$. According to the Removable Singularity Theorem of Riemann, the so defined function $g(t)$ is holomorphic in the vicinity of t_0. □

Now suppose that a function $f(t)$ is holomorphic and elementary. The function $g(t)$ is surely elementary at points $t \neq t_0$ being a regular superposition of elementary functions. Then what can be said about elementariness of $g(t)$ at $t = t_0$?

ANSWER: Nothing. Function $g(t)$ may be either elementary or not at $t = t_0$.

EXAMPLE 2. *Consider* $f(t) = \sin t$ *at* $t = 0$. *Then* $g(t) = \dfrac{\sin t}{t}$ *being not elementary at* $t = 0$ *(among several other such function in Table 1).*

EXAMPLE 3. *Consider* $f(t) = \sqrt{t+1}$ *at* $t = 0$. *Then* $g(t) = \dfrac{\sqrt{t+1}-1}{t}$ *and this function was earlier demonstrated to be elementary* $t = 0$ *satisfying a regular IVP*

$$g' = -\frac{g^2}{2tg+2}, \qquad g|_{t=0} = \frac{1}{2}.$$

Function $g(t)$ *is an algebraic function satisfying a polynomial*

$$tg^2 + 2g - 1 = 0$$

having $t = 0$ *as a critical point and* $t = -1$ *as a singular point.*

E-mail address: alex@TaylorCenter.org
URL: http://TaylorCenter.org