Les dances dels N cossos

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The N-Body Problem

Let $z_j \in \mathbb{R}^2$, $j = 1, \ldots, N$ the positions of the bodies and $m_j > 0$, $j = 1, \ldots, N$ the masses. We shall mainly refer to the **planar problem**.

Equations of motion under gravitational attraction

$$\ddot{z}_j = \Sigma_{k=1,\, k\neq j}^{N} m_k \nabla_{z_j} f(r_{k,j}) \ ,$$

where $r_{k,j} = |z_k - z_j|$ and $f(r)$ is the **two-body potential** equal to $1/r$ in the **Newtonian case**. Later on we shall comment on the **strong force potentials** $f(r) = 1/(ar^a), a \geq 2$.

The problem has **fist integrals**:

1) the **center of mass** $\Sigma_{k=1}^{N} m_k z_k$ moves on a straight line with constant velocity. From now on we take $\Sigma_{k=1}^{N} m_k z_k = 0$,

2) the **angular momentum** $c = \Sigma m_k z_k \wedge \dot{z}_k$,

3) the **energy** $H = K + U$, where $K = \frac{1}{2}\Sigma_{k=1}^{N} m_k |\dot{z}_k|^2$ (**kinetic energy**) and $U = -\Sigma_{1\leq k<j\leq N} m_k m_j f(r_{k,j})$ (**potential energy**).

It is also important to use the **moment of inertia** $I(z) = \Sigma_{k=1}^{N} m_k (z_k, z_k)$.

Solutions

Unless we consider some **very special solution** only the **two-body problem** can be solved **explicitly**.

The simplest solutions would be **fixed points**. There are not, but they can be found if we consider the problem in **rotating coordinates**.

They give rise to the so called **relative equilibrium solutions (res)**. The N bodies rotate as if they were a **rigid body**.

They can be found by requiring $\ddot{z}_j = \lambda z_j, j = 1, \ldots, N$, where λ is a constant, the same for all j, or as **critical points** of **the potential U restricted to some given level of the moment of inertia**: $U|_{I=M\rho^2}$, ρ being the **radius of inertia** and $M = \Sigma_{k=1}^{N} m_k$.

If we consider the case of **equal masses** $m_j = 1, j = 1, \ldots, N$,

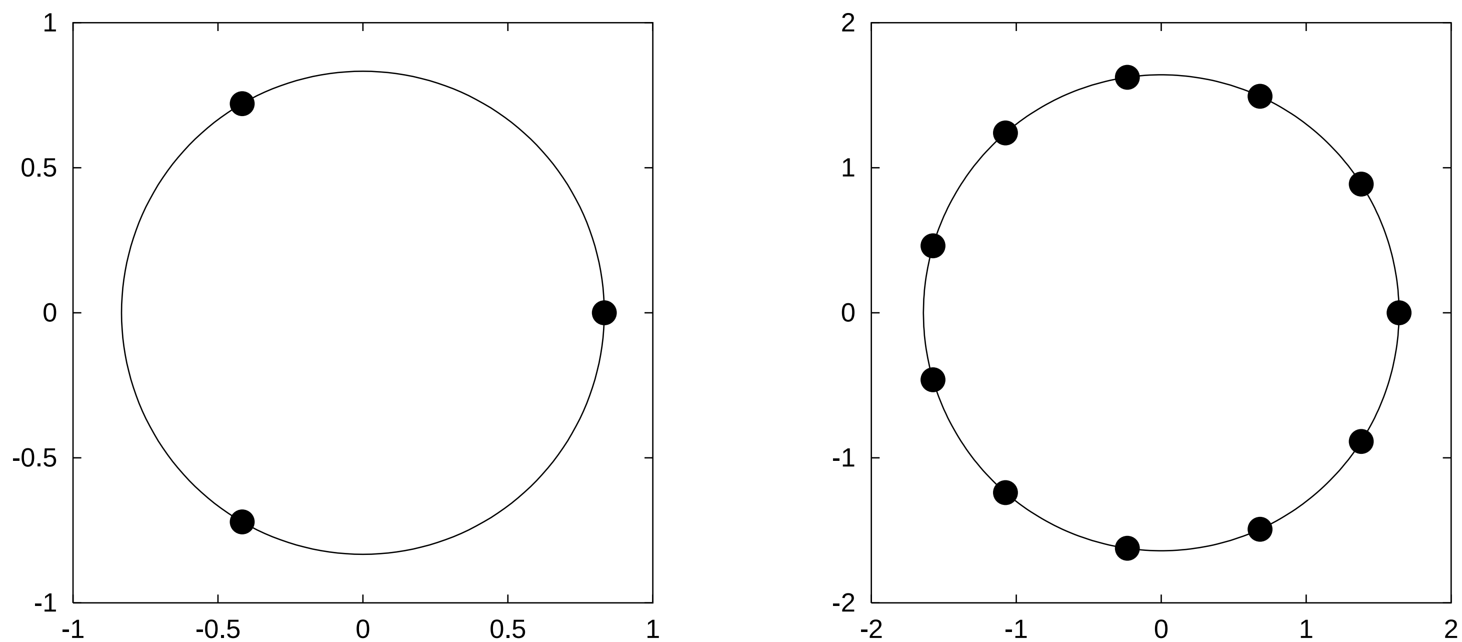
as we shall do in what follows,

a **regular** N-gon is a **res** with **all bodies moving periodically on the same circle**. This kind of solution for $N = 3$ was discovered by **Joseph-Louis Lagrange (Torino 1736-Paris 1813) in 1772**. (Born as Giuseppe Ludovico Lagrangia).

Due to the **homogeneity** one can **scale time and distance** so that it is enough to consider the solutions: a) on a given **level of energy** $h < 0$, or b) on a given **level of inertia** $I = \rho^2$, or c) with a **fixed period**.

For most of the talk we shall consider the **period fixed:** $T = 2\pi$.

Then the radius, R, of the circle circumscribed to the N-gon is given by $R = \frac{1}{2R^{a+1}}\sigma_{a,N}\,,$ where $\sigma_{a,N} = \Sigma_{j=1}^{N-1}(2\sin(j\pi/N))^{-a-1}\,.$



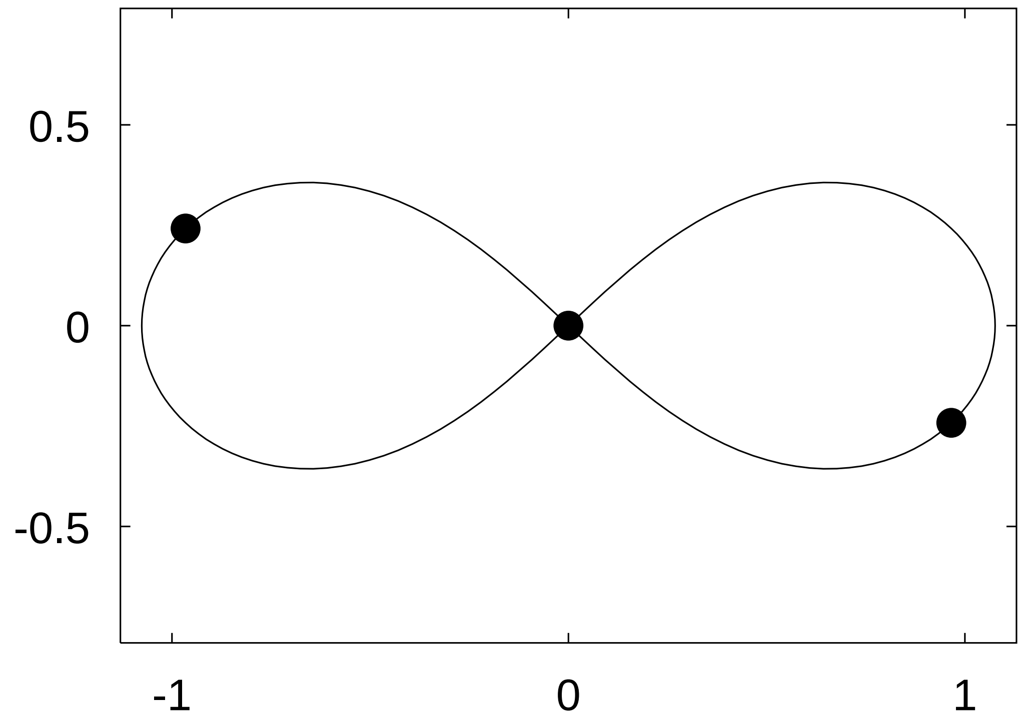
Note in these examples ($N = 3, N = 11$) that **all the bodies move periodically on the same circle**.

A natural question

Are there other **periodic solutions** such that all bodies with **equal masses** move on the plane along **the same path**?

Only a few years ago a solution in the **Newtonian case** with 3 bodies on the same planar curve, different form a circle, has been proved to exist by Chenciner and Montgomery (December 1999). Moore found also the same orbit in a previous numerical work in 1993 in a different context.

This curve is a **figure eight curve**.



Sketch of the existence proof

Is based on the **minimization of the action functional** (see later). One can take as **test path a curve with U constant** (equal to the value of U at the collinear configuration), inside I constant, that one travels with **constant velocity**. A strong use is made of the symmetries. It is checked, analytically, that an optimal choice of I allows **to rule out the possibility of both triple and binary collisions**. The proof requires **one piece of numerical information**: The evaluation of a definite integral along some path defined implicitly. See Chenciner–Montgomery for details.

References

Chenciner, A. and Montgomery, R.: A remarkable periodic solution of the three body problem in the case of equal masses, *Annals of Mathematics* 152, 881–901 (2000).

Moore, C.: Braids in Classical Gravity, *Physical Review Letters* 70, 3675–3679 (1993).

Some properties of the figure eight solution with $N = 3$

a) **It passes through all collinear configurations**. When 1/12 of a period is known, from collinear to isosceles, the full curve is obtained by the symmetries. **The angular momentum is zero**.

b) I and U are **almost constant**: $I \in [1.973, 1.982]$, minimum at isosceles, maximum at collinear; $U \in [2.511, 2.667]$, minimum at collinear, maximum at isosceles.

c) The curve is quite **close** to an affine transformation of a **lemniscate**. A fit by a polynomial (in (x, y)) of degree 4 gives errors of the order of 10^{-4}, and they are of the order of 10^{-7} when degree 8 is used.

d) The **eigenvalues** of the monodromy matrix, beyond the trivial ones, are $\exp(\pm 2\pi i \nu_j)$, $\nu_1 \approx 0.00842272$, $\nu_2 \approx 0.29809253$. Hence, it is **linearly stable**.

e) It is possible to obtain an **analytical expression of a Poincaré map** around the fixed point, with the coefficients **computed numerically**. A routine normal form check gives that **the torsion is indefinite**. Hence **KAM theorem applies**. 3D invariant tori exist around the figure eight.

f) Some **“satellites”** of the figure eight give also periodic solutions such that the three bodies are **on the same path**, but this is **not true** for all periodic satellites. (See illustrations).

g) It is possible to **continue** the figure eight periodic solution to $c \neq 0$. It produces a periodic solution in **rotating coordinates** keeping planar motion. This produces also solutions (**in fixed axes**) with the three bodies on the same curve by choosing suitable values of c.

h) The eight can be **continued to all** $a > 0$ (the exponent in the potential) and even to $f(r) = \log r$ and beyond. It is found to be linearly stable only for $(1.228\ldots > a > 0.868\ldots)$.

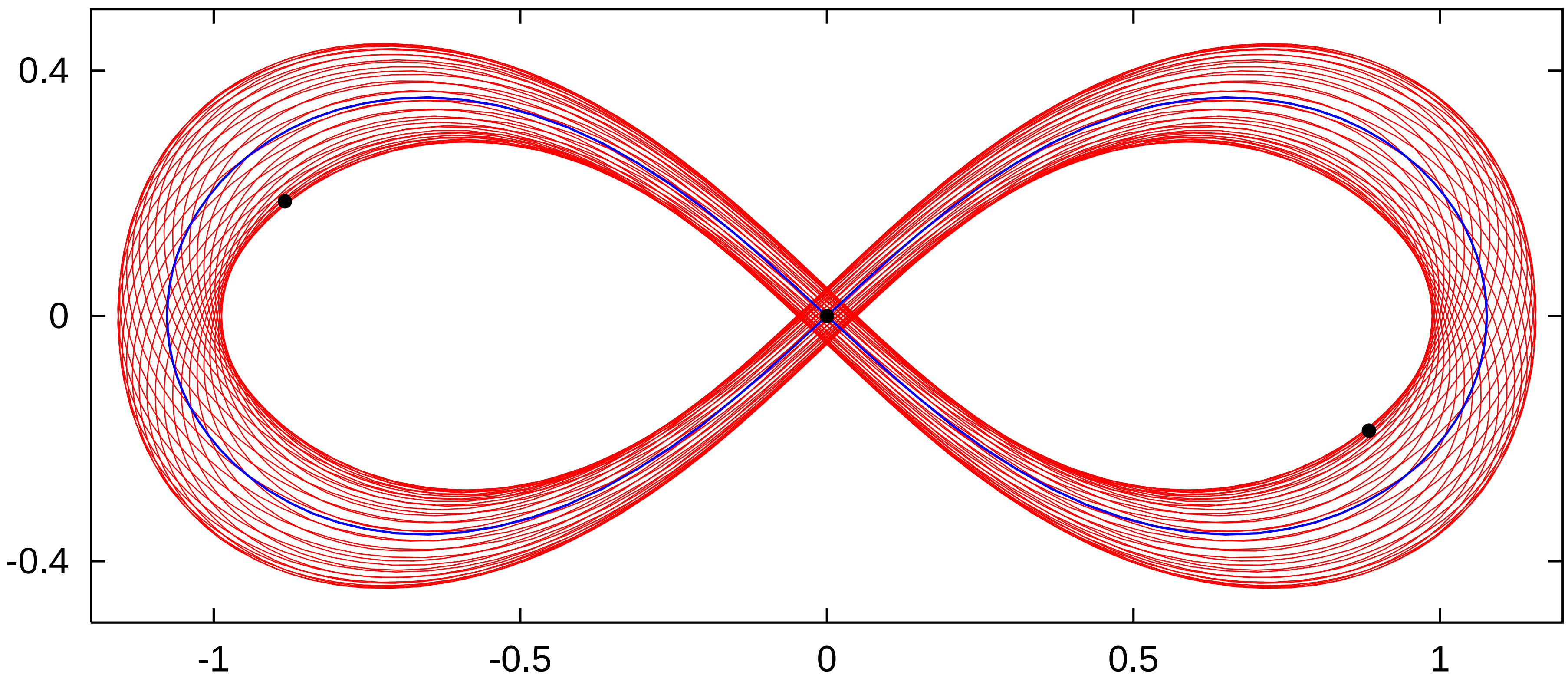
i) It is possible to **continue** the periodic solution to other **nearby masses**, each moving then in a **slightly different “figure eight”**. Stability is only preserved for relative variations of the order of 10^{-5}.

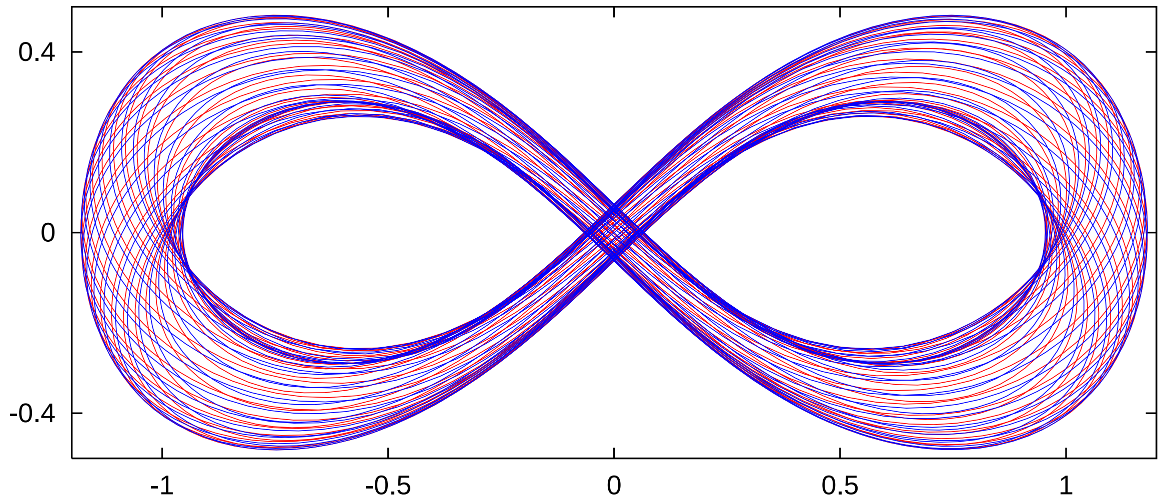
j) Two more families **bifurcate** when changing the **horizontal components of the angular momentum**. Along one of them one finds the **Lagrange solution traveled twice** (Chenciner, Féjoz, Montgomery, *Nonlinearity*, 2004). That is, there is a family of periodic solutions **joining** the eight with the equilateral solution.

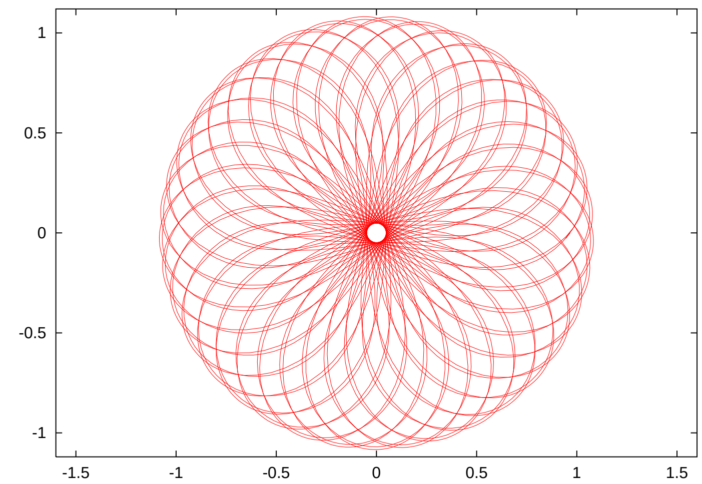
A reference

Simó, C.: Dynamical properties of the figure eight solution of the three–body problem, *Contemporary Math. AMS* 292, 209–228 (2001).

A **"satellite"** orbit of the figure eight. Only the **fast mode is excited**. **Rotation number 11/37**. The three bodies **travel on the same path**. For reference also the figure eight orbit is shown.







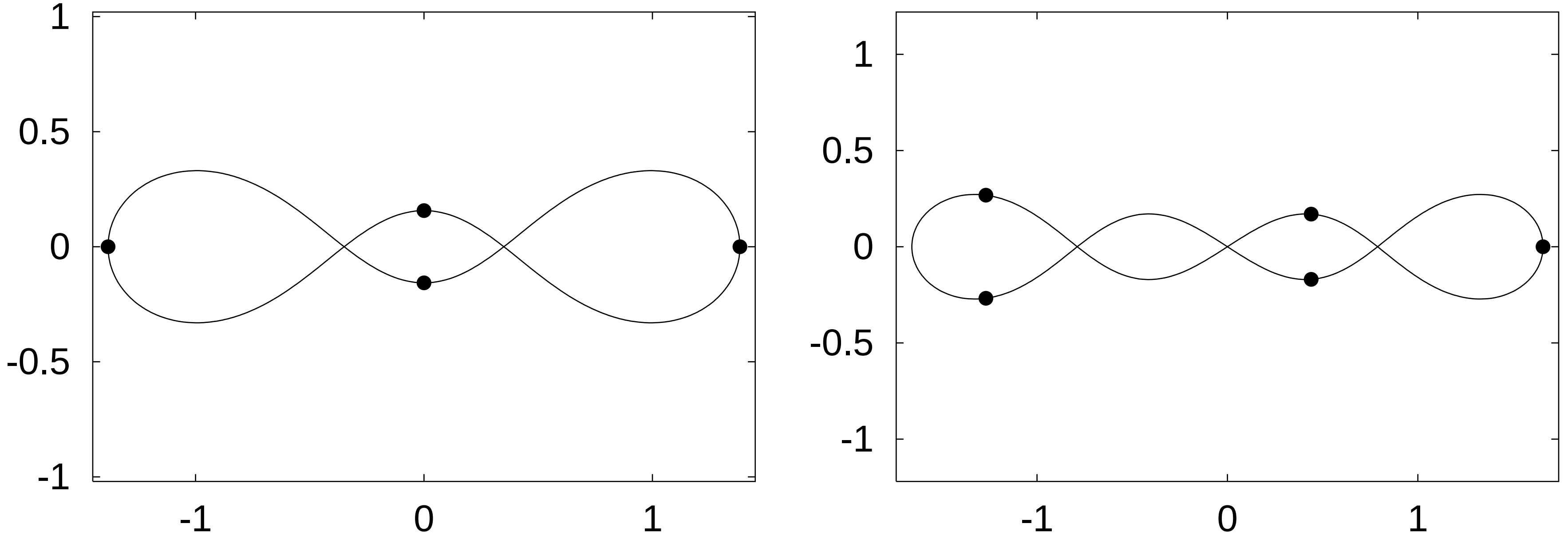
Left: A **"satellite"** orbit of the figure eight. Only the **fast mode is excited**. **Rotation number 8/27**. The three bodies **travel on different paths**. Only two of them are shown here.

Right: A **"relative choreography"**. By taking $c \neq 0$ one finds **choreographies in rotating axes** (suggested by M. Hénon). After one period in the rotating frame has **rotated δ in a fixed frame**. If $\delta \in 2\pi\mathbb{Q}$ we get a **choreography in the fixed frame**. In this case $\delta = \frac{3}{37} 2\pi$.

Beyond $N = 3$

At the end of 1999 Gerver found a **"supereight"** solution with $N = 4$.

It was a simple exercise to find a **huge amount of solutions** with all the bodies in the **same curve** and with quite **different shapes** of the curves. Initially, in the Newtonian case. Later on, with different potentials.

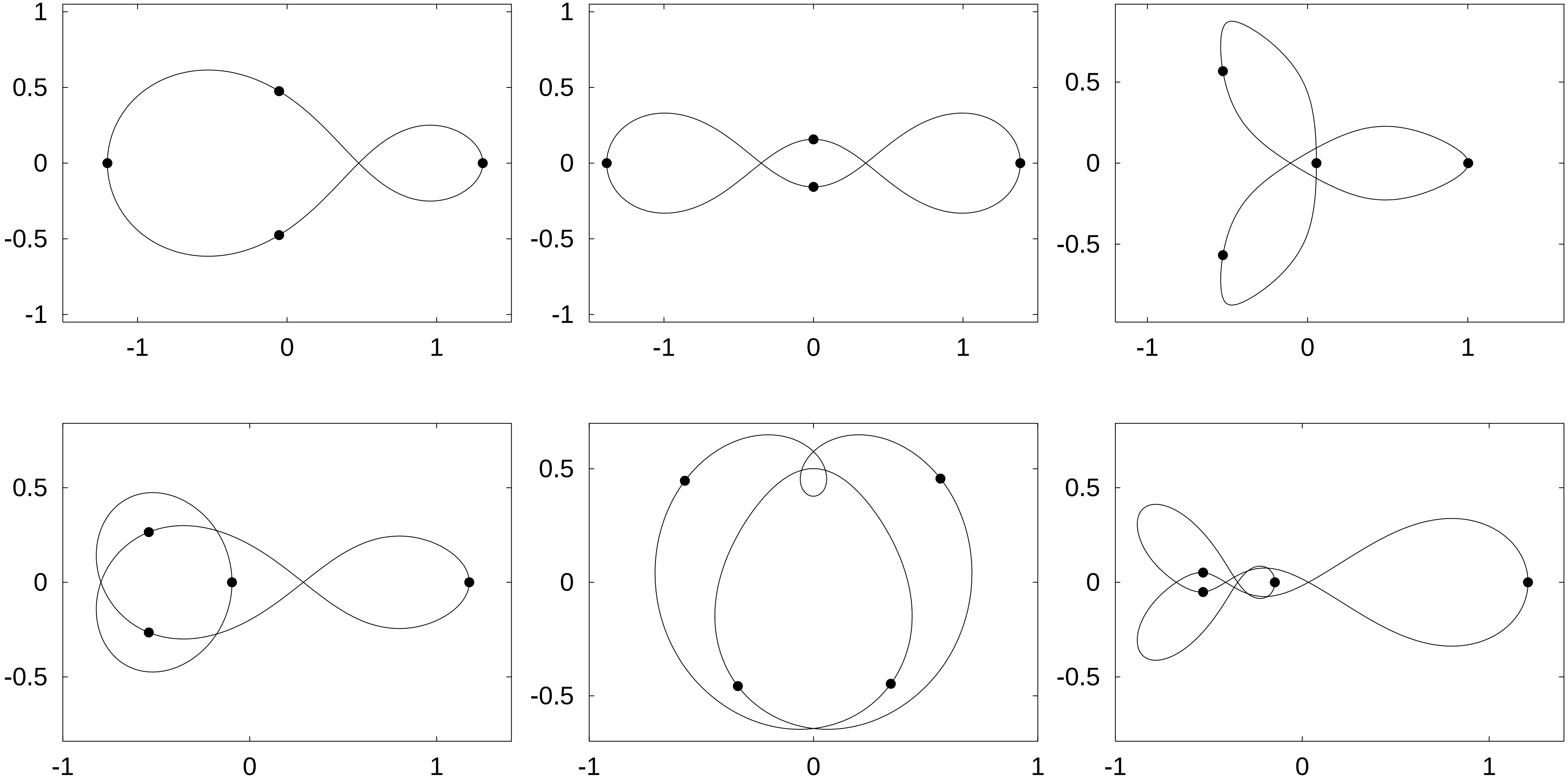


I named them **choreographies** because of the **dancing-like motion** of the bodies seen in animations, as we will see later.

(Rather **simple choreographies** because they are on the **same curve**. ***k*-choreographies** should be used for bodies moving on k **different curves**).

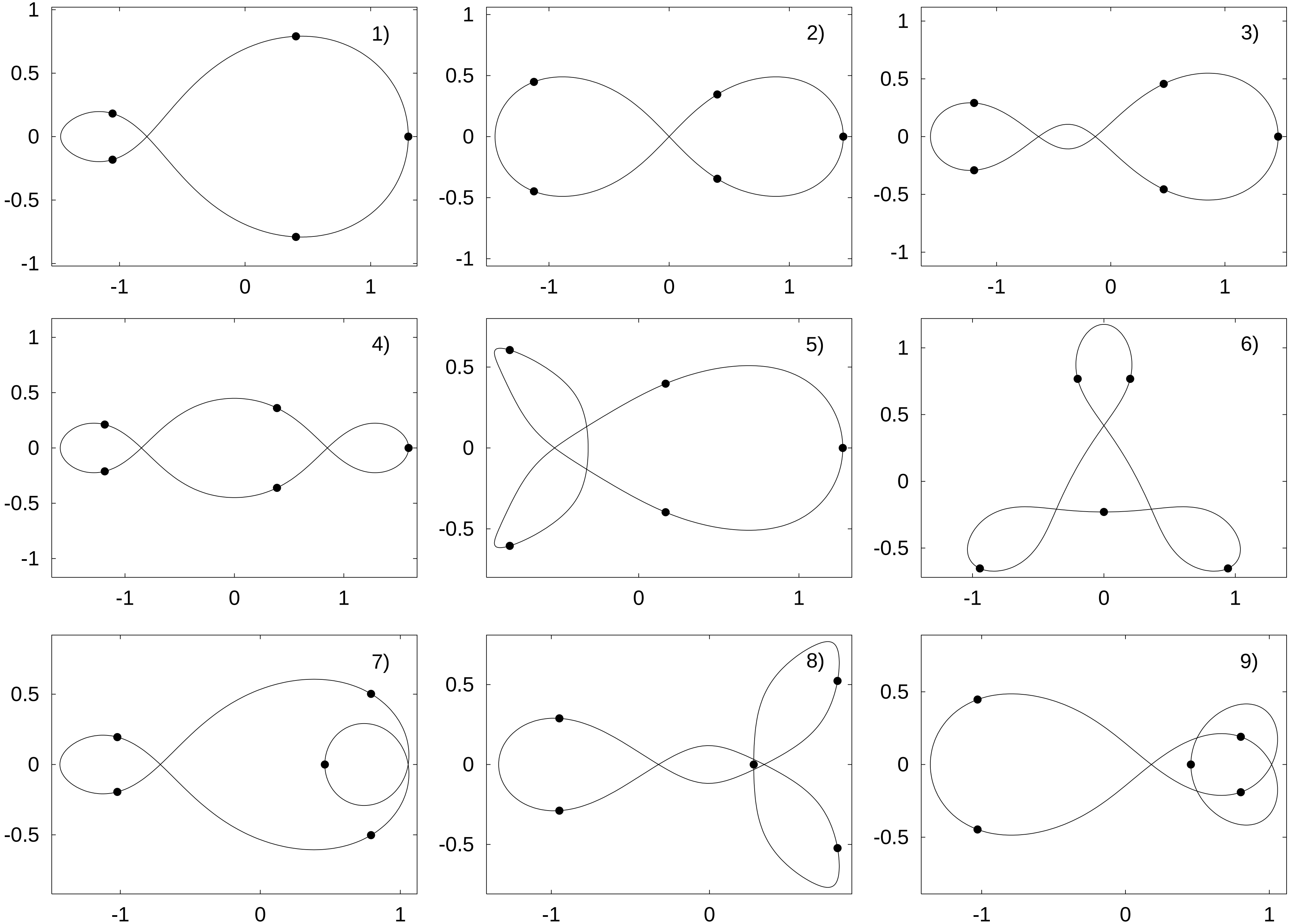
Two choreographies which differ only by **change of scale, rotation, change of orientation, symmetry, etc**, will be seen as **the same**.

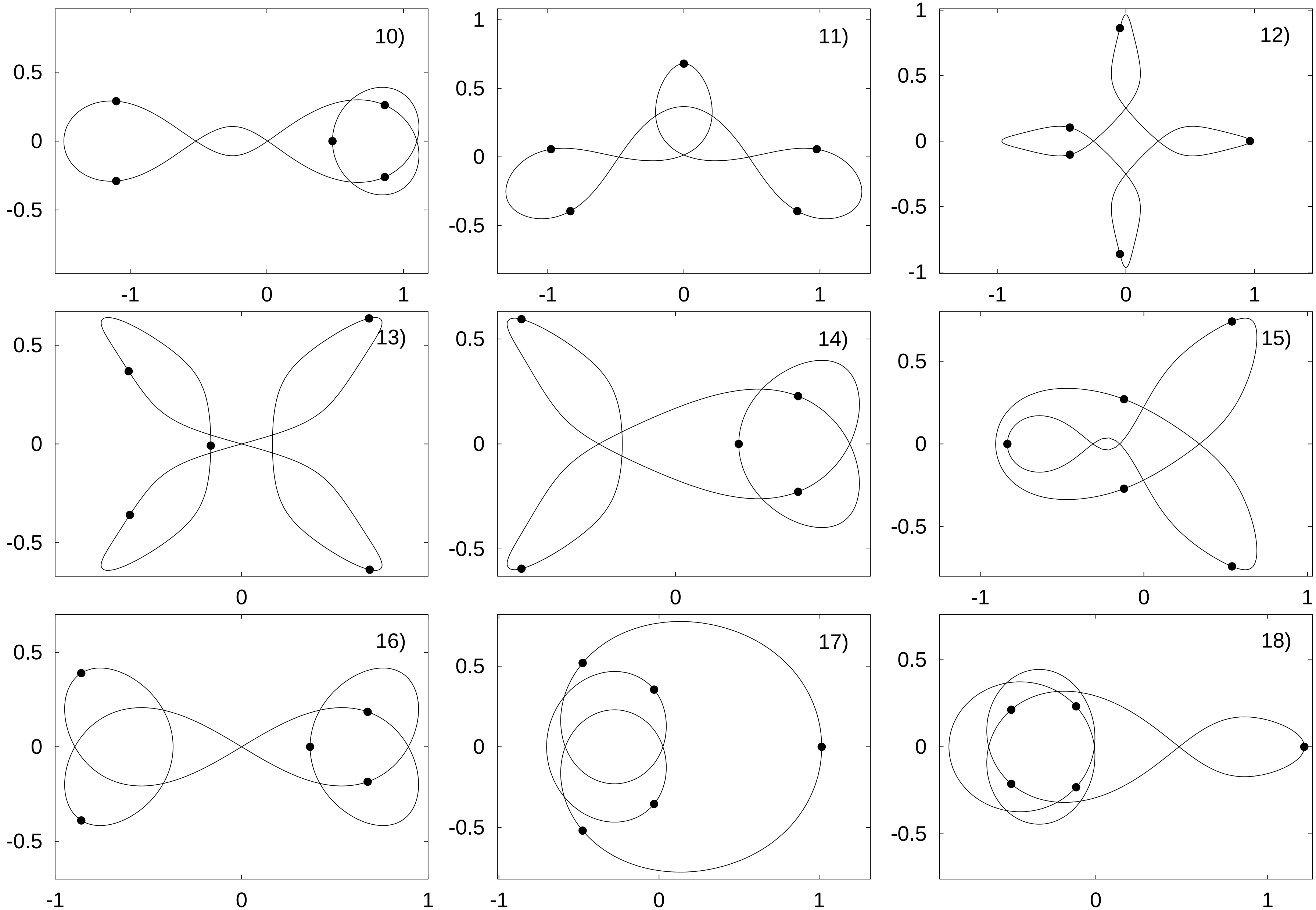
A sample of choreographies for $N = 4$ **is presented**. Newtonian case.



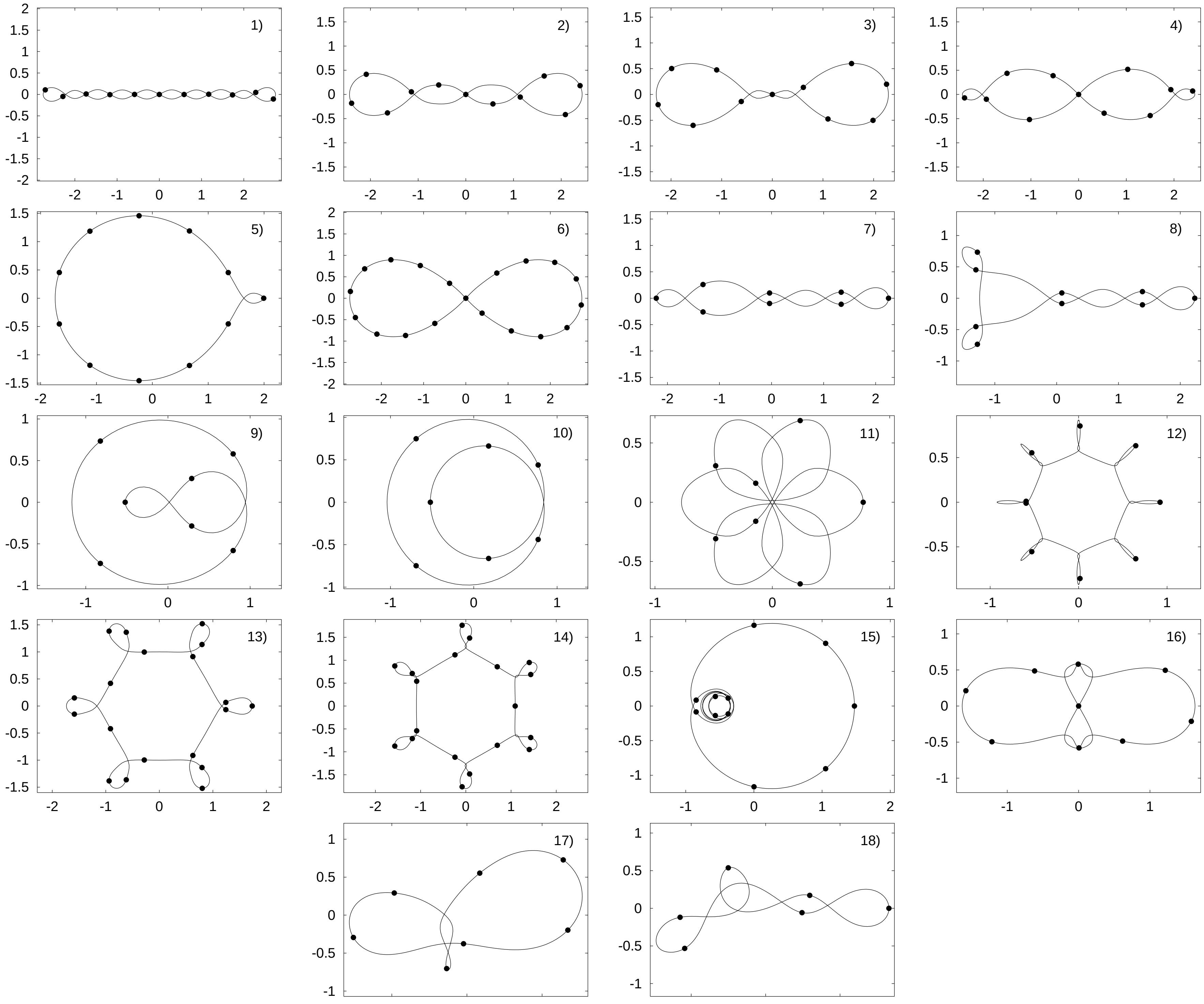
And now some examples with $N = 5$. Most of them can be seen as **linear chains** with **loops of different size**. Some loops are eventually **folded**.

Number 1 consists of a **large loop and a small one**. In the small loop there are **either 1 or 2 bodies for all t**.





Note that for $N = 4, 5$ no solution with **one small inner loop** has been found in the Newtonian case. Next we see some **additional examples** and **a movie with several choreographies**.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The set up of the problem

We look for 2π–**periodic functions** $q : \mathbb{S}^1 \mapsto \mathbb{R}^2$ such that if

$$z_j(t) = q(t - (j-1)2\pi/N)\,, \quad j = 1, \ldots, N,$$

we find a **solution** to the equations of motion.

Different approaches:
i) The **variational** approach: **Minimize** (or, in general, make extremal) **the action** $A = \int_0^{2\pi} L(t)dt$, where $L = K - U$ (the **Lagrangian**) and $K = K(\dot{z}_1, \ldots, \dot{z}_N)$, $U = U(z_1, \ldots, z_N)$. This is equivalent to **minimize**

$$\int_0^{2\pi/N} L(q(t), \ldots, q(t + (N-1)2\pi/N), \dot{q}(t), \ldots, \dot{q}(t + (N-1)2\pi/N))dt\,.$$

ii) The **flow** approach: Look for **initial data**

$$z_1(0), \ldots, z_{N-1}(0), z_N(0), \dot{z}_1(0), \ldots, \dot{z}_{N-1}(0), \dot{z}_N(0)$$

such that under $\Phi_{2\pi/N}$, where Φ_t denotes the **time-t flow of the N-body problem**, one obtains $z_2(0), \ldots, z_N(0), z_1(0), \dot{z}_2(0), \ldots, \dot{z}_N(0), \dot{z}_1(0)$.
iii) It is also possible to look for $q(t)$ as the solution of a **differential delay equation**.

Some practical comments

For **numerical computation** of choreographies, **both i) and ii) are used**.

Severe **difficulties** appear in **highly unstable periodic orbits** or in **orbits passing close to collision**.

It is needed to use **parallel shooting**. It allows to compute periodic orbits even with **dominant eigenvalue of the monodromy matrix** larger than 10^{100}.

As a **general procedure** it is very efficient

1) To start a **variational approach with strong force potential** (a=2) to have an initial approximation,

2) To **refine** it by using the **flow method** solving for $z_i(0), \dot{z}_i(0), i = 1, \ldots, N$ using **Newton method**, and

3) To do **continuation** (using the flow approach) with respect to the **exponent in the potential**, trying to reach $a = 1$, **if it is possible**.

In 2) it is required to **compute the first variational equations** and this gives the **stability properties** as byproduct.

The functional space

The suitable space where we look for solutions is the **Sobolev space** $H^1(\mathbb{S}^1, \mathbb{R}^2)$ (or H^1 for shortness) of functions with **square integrable first derivative.**

The difficulties: Problems appear when the solution approaches a **collision**.

A **collision** occurs if there exists a **double point** $q(t_1) = q(t_2)$, $t_2 - t_1$ multiple of $2\pi/N$. Let $\Delta \subset H^1$ be the **functions associated to collisions**.

We would like to see that **in each connected component** of $H^1 \setminus \Delta$

(or **choreographic class**)

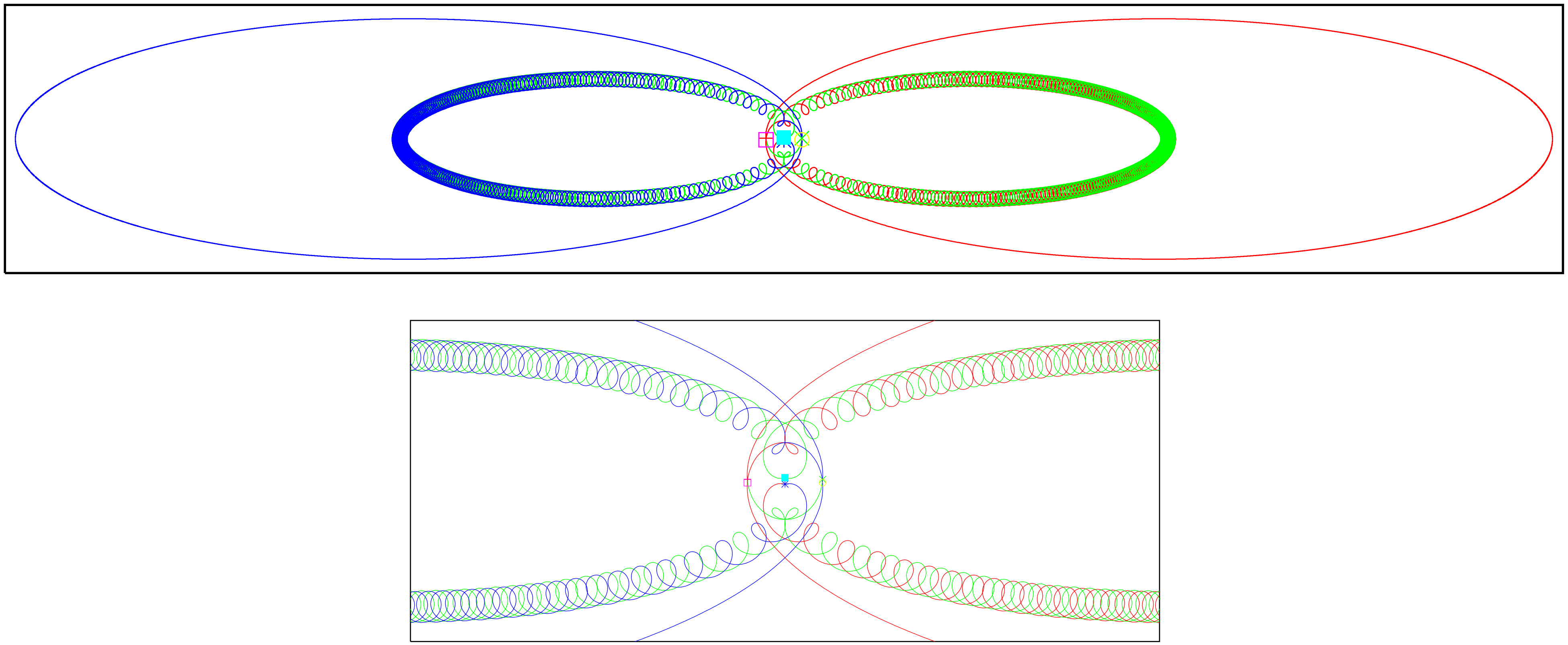
there is a solution minimizing the action. Unfortunately, this seems **not to be true** for the Newtonian potential. However

Theorem: Consider the case of a strong force potential as defined above ($a \geq 2$). Then in every choreographic class there is a solution minimizing the action A.

The main reason is that $\forall a < 2$ the **contribution of a collision to A is bounded** while for $a \geq 2$ **it becomes unbounded**.

How many choreographies for $N = 3$?

Question: whether the **number**, even for $N = 3$, is **finite or not**. The answer is **NOT**. It requires a **Computer Assisted Proof (CAP)** involving rigorous estimates on the so-called **invariant weakly hyperbolic manifolds of invariant objects at infinity**.



Top: A **choreography** of the 3-body problem. Bottom: A magnification of the central part. In each one of the **binary portions**, the bodies in the binary make **200 revolutions around their centre of masses**.

Some additional references

Simó, C.: New families of Solutions in N–Body Problems, in *Proc. 3rd Euro. Cong. Math., Progress in Math. 201*, 101–115, Birkäuser, 2001.

Chenciner, A., Gerver, J., Montgomery, R., Simó, C.: Simple Choreographic Motions of N Bodies: A Preliminary Study, in *Geometry, Mechanics, and Dynamics*, ed. P. Newton et al., 298–309, Springer, 2002.

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