A remarkable periodicity in a real valued extraction of a well known complex function

Alexander Gofen, Stephen Lucas, James Sochacki
alex@taylorcenter.org, lucassk@jmu.edu, sochacjs@jmu.edu

Keywords and phrases: ODE, Complex functions, periodic.
ZDM Subject Classification: I20,I70,R20.

Abstract We deal here with a complex quadratic ODE with distinct equilibria. The solution is determined by an integral and Möbius transformations. By extracting a particular real valued function out of this complex elementary function, we were able to discover some remarkable properties of the dynamics of the real trajectory. Namely, with an independent variable t understood as time, it takes the same time interval to trace every circular orbit or every lap of the spiral trajectory (independent of the size of the circle or spiral determined by the solution). These remarkable properties were first discovered by observation of the real time drawing of such trajectories in an advanced ODE solver called the Taylor Center.

1. INTRODUCTION.

We consider the complex quadratic ODE

$$z' = Az^2 + Bz + C = \mu(z-p)(z-q) \tag{1.1}$$

when the roots p and q are distinct, and all coefficients are complex. Here the independent variable $w = t + is$ and $z = z(w) = x(t,\ s) + iy(t,\ s)$. The text by Hirsch, Smale and Devaney [**5**] has this problem as an exercise, but the authors probably did not know the properties of this ODE that will be presented in this article. They wanted the students to analyze the equilibrium solutions which we do not do in this article, but we do mention what type of equilibium solutions there are. Using a composition with a Möbius transform ([**1**]), helped us determine these properties after a detailed numerical study using the software Taylor Center [**4**]. Taylor Center is an ODE software package that can generate high degree Taylor polynomial approximate solutions to an arbitrary ODE using automatic differentiation. We detail how the reader can obtain a copy of Taylor Center and experiment with sample ODEs in Appendix 2. (See Gofen [**2, 3**] for a description of automatic differentiation.) All the figures presented here were generated in Taylor Center. We also discovered that highly accurate numerical solutions are needed to present correct visualization of the solution to this ODE.

Following the path set in [**6**], we consider a particular method of a real valued extraction[1] from the complex solution $z = z(w)$ comprised of its components

[1] The method of extraction of a real valued trajectory used here is a first one that comes to mind, but only one of infinitely many ways such an extraction may be done. In the independent variable $w = t + is$, we set $s = 0$. However we could choose some parametric curve $(t(\tau),\ s(\tau))$, or any line in the complex plane to extract other real valued trajectories $(x(\tau),\ y(\tau))$.

$x(t,\ s)$ and $y(t,\ s)$ under the assumption that $s \equiv 0$ (meaning that the independent $w = t$ varies along the real axis only). The obtained real trajectory $(x(t),\ y(t))$ satisfies the respective system of real valued ODEs derived from Equation (1.1)

$$\begin{aligned} x' &= P(x,\ y) = a_r x^2 - a_r y^2 - 2\, a_i xy + b_r x - b_i y + c_r \\ y' &= Q(x,\ y) = a_i x^2 - a_i y^2 + 2\, a_r xy + b_r y + b_i x + c_i \end{aligned} \tag{1.2}$$

where all of the coefficients are real (the independent variable $t' = 1$). The roots p and q of Equation (1.1) are the equilibria; either both are centers, or one equilibrium is stable while the other is unstable. The polynomials P, Q have the form shown, but the coefficients have special properties for the P, Q we will consider below.

Using these ODEs and the fact about the roots, Sochacki and Lucas [**6**] studied several types of real trajectories which happened to be circles or double spirals. An example is shown in Figure 1. Here we report on recently discovered remarkable properties exhibited by these trajectories and other trajectories for Equation 1.1.

Namely, with independent variable t understood as time, it takes the same time interval to trace every circular orbit or every lap of the spiral trajectory $(x(t),\ y(t))$ independent of size. This remarkable property was first observed in the real time drawing of such trajectories in the advanced ODE solver called the Taylor Center [**4**] and then shown using this software to be numerically true. Studying and proving this special kind of periodicity is the goal of this paper.

Observing this phenomenon in animation is possible. However, viewing this in the phase portrait is not possible and, in fact, not intuitive. If one looks at the four trajectories with initial conditions (0.1,0),(-0.1,0),(0,0.1) and (0,-0.1) for the same ODE that generated the solution curve in Figure 1, one sees in Figure 2 that the laps in the spirals for each of these have significantly varying sizes even though the initial conditions are relatively close. In Figure 3 the radius of the largest lap in the black spiral (initial condition (0.1,0)) is 18, the radius of the largest lap in the blue spiral (initial condition (-0.1,0)) is 80, the radius of the largest lap in the red spiral (initial condition (0,0.1)) has radius 25, and the radius of the largest lap in the green spiral (initial condition (0,-0.1)) has radius 10,000 (this largest lap in the spiral is not shown because of the three orders of magnitude larger than the other spirals). One would not expect from Figure 3 that all these laps take the same period to be traced out. Also, unless one uses a highly accurate method like AD one will not get an accurate representation of the larger laps in these spirals. We will discuss the 'sensitive dependence on initial conditions' seen in these two figures more in Appendix II.

Remark 1. *If in Equation (1.1) $z = x + yi$ and $w = t + si$ are chosen with $y = 0,\ s = 0$, and $A,\ B,\ C$ are real, then we obtain the real ODE*

$$x' = Ax^2 + Bx + C$$

whose solutions are formed from

$$x = \tan(\alpha t + \beta),$$

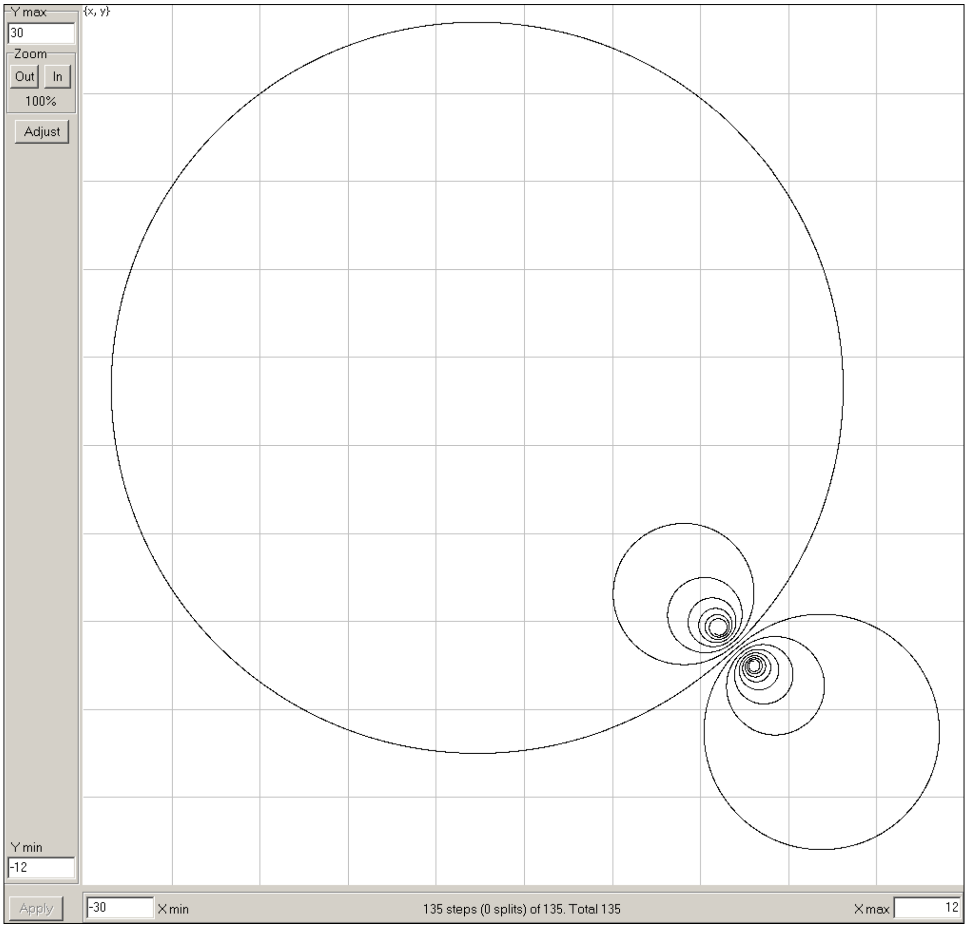

FIGURE 1. A Double spiral with a declined axis as it was first encountered.

$$x = \frac{x_0}{1 \pm x_0 t},$$

or

$$x = \frac{(x_0 - r)e^{2\,tr} + x_0 + r}{(r - x_0)e^{2\,tr} + x_0 + r}$$

depending on whether the ODE has no equilibria, one equilibrium or two equilibria.

2. The setting

The general solution $z(w)$ of the ODE (1.1) satisfies the condition

$$\mu(p-q)\,(w + const) = \ln \frac{z-p}{z-q} \tag{2.1}$$

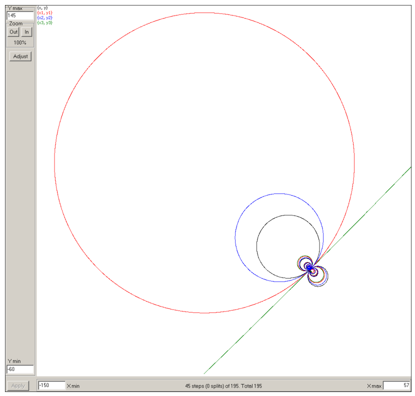

FIGURE 2. Four initial double spirals.

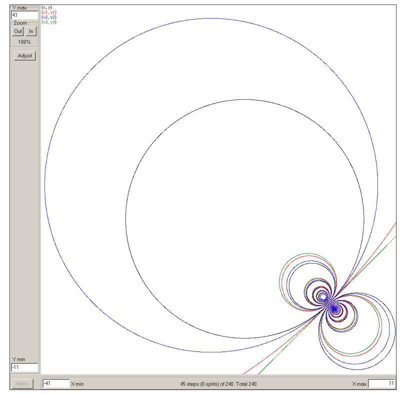

FIGURE 3. The full solution curves for Figure 3.

In order to figure out and prove the above mentioned periodicity we will obtain the explicit formula for $z = z(w)$ from the general solution (2.1) and extract the real valued parametric representation $(x(t),\ y(t))$.

The complex roots $p \neq q$ of the quadratic polynomial in Equation (1.1) are stationary points for the solutions of Equations (1.1) and (1.2) meaning that if the solution is a spiral, it winds around these roots (Figure 4). The line connecting p and q will be referred as the axis of the double spiral, and generally the axis is declined. We define the laps of the double spiral as the 360° pieces of the spiral cut by this axis.

3. The case

$q = i, \quad p = -i$

For the purpose of simplicity we are considering only the case when the spiral is perpendicular to the x axis and, furthermore, $p = -i,\ \ q = i$. (This corresponds to the complex tangent.) However, all solutions to Equation (1.1) can be found from magnifications, rotations and/or translations of this case. The general solution with these roots takes the form

$$e^{-2\mu i(w+c)} = \frac{z+i}{z-i} \tag{3.1}$$

$$(z-i)e^{-2\mu i(w+c)} - z - i = 0$$

$$z = i\frac{e^{-2\mu i(w+c)}+1}{e^{-2\mu i(w+c)}-1}.$$

CONCLUSION 1. *The subclass of the quadratic ODEs with* $p = -i, \quad q = i$ *in the general solution (2.1) is*

$$z' = \mu(z^2+1), \qquad z = x+iy, \qquad \mu = a+ib \neq 0, \tag{3.2}$$

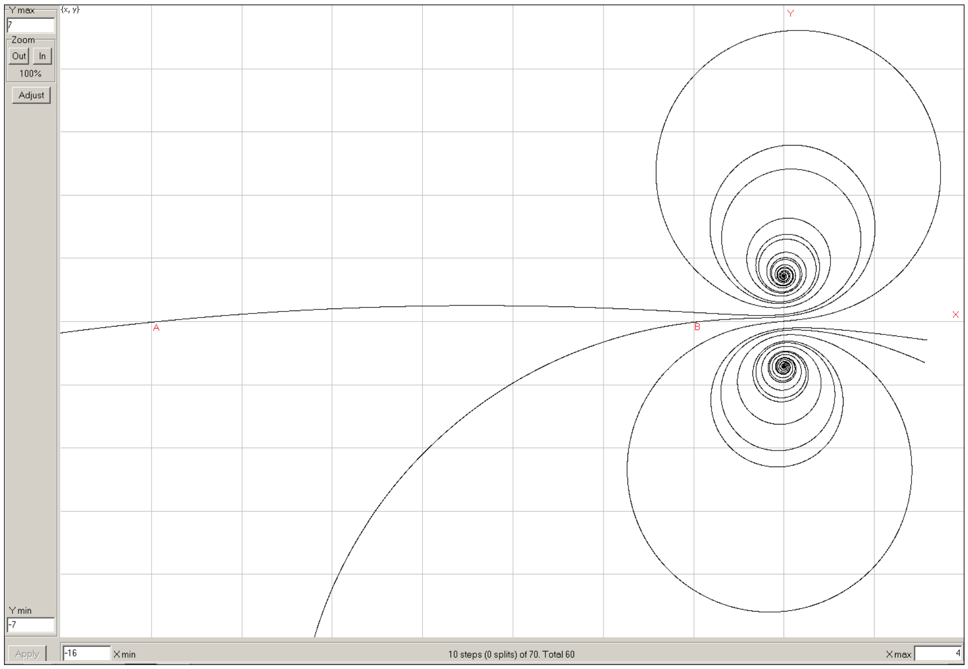

FIGURE 4. Spiral Portrait.

and in real coordinates is

$$
\begin{aligned}
x' &= a(x^2 - y^2 + 1) - 2bxy \\
y' &= b(x^2 - y^2 + 1) + 2axy.
\end{aligned} \tag{3.3}
$$

Denote $u = \mathrm{Re}(-2\mu i(w+c))$, $v = \mathrm{Im}(-2\mu i(w+c)$, and transform formula (3.1) into

$$
\frac{e^{-2\mu i(w+c)}+1}{e^{-2\mu i(w+c)}-1} = \frac{e^{u+iv}+1}{e^{u+iv}-1} = \frac{e^{u}(\cos v + i\sin v)+1}{e^{u}(\cos v + i\sin v)-1} =
$$

$$
\frac{e^u\cos v + 1 + ie^u \sin v}{e^u \cos v - 1 + ie^u\sin v} = \frac{[(e^u\cos v+1)+ie^u\sin v]\,[(e^u\cos v - 1) - ie^u \sin v]}{e^{2u}\cos^2 v - 2e^u\cos v + 1 + e^{2u}\sin^2 v} =
$$

$$
\frac{e^{2u}\cos^2 v - 1 + e^{2u}\sin^2 v + i(e^{2u}\cos v\sin v - e^u\sin v - e^{2u}\cos v \sin v - e^u \sin v)}{e^{2u} - 2e^u\cos v + 1}
$$

$$
= \frac{e^{2u} - 1 - 2ie^u\sin v}{e^{2u} - 2e^u \cos v + 1}
$$

so that

$$
\begin{aligned}
x(t,\ s) &= \frac{2e^u\sin v}{e^{2u} - 2e^u\cos v + 1} \\
y(t,\ s) &= \frac{e^{2u}-1}{e^{2u} - 2e^u\cos v + 1}.
\end{aligned} \tag{3.4}
$$

This is a parametric solution for the ODE (3.2).

Conclusion 2. *The denominator* $e^{2u} - 2e^u \cos v + 1 \geq 0$ *because*

$$e^{2u} - 2e^u \cos v + 1 = (e^u - 1)^2 + 2e^u(1 - \cos v)$$

meaning that it reaches zero only when $u = 0$ *and* $v = 2n\pi$.

Recalling what u, v, and μ are, we obtain

$$\begin{aligned} u(t,\ s) + iv(t,\ s) &= -2i\mu(w + c) = -2i(a + bi)(t + c_r + i(s + c_i)) \\ &= -2i(at + ac_r - bs - bc_i + i(bt + as + ac_i)) \\ &= 2(bt + as + ac_i) - 2i(at + ac_r - bs - bc_i) \end{aligned}$$

and with $s = 0$

$$\begin{aligned} u(t) &= 2(bt + ac_i) \\ v(t) &= -2(at + ac_r - bc_i). \end{aligned} \tag{3.5}$$

As $\mu = a + bi \neq 0$ so that a and b cannot both be zero, we have the following cases

Case 1. (Closed circles) *If* $b = 0,\ a \neq 0$ *so that* $u = 2ac_i = const$ *but* $v(t)$ *varies, both* $x(t)$ *and* $y(t)$ *are periodic (due to periodicity of* $\sin v$ *and* $\cos v$ *in* t*), meaning that the trajectory* $(x(t),\ y(t))$ *is a closed curve as shown in Figure 5 in black. Later, we will show this closed curve is a circle. In particular, the zeros of* $x(t)$ *are periodic meaning that it takes the same time to trace each circle for every diameter .*

Case 2. (Two Single or one Double spiral) *If* $b \neq 0$ *and* $a \neq 0$ *meaning that both* u *and* v *vary, neither* $x(t)$ *nor* $y(t))$ *is periodic and the trajectory is not closed, but zeros of* $x(t)$ *are periodic (due to periodicity of* $\sin v$ *in formula (3.4)). This implies that* it does take the same time to trace every lap of the spiral. *(This is what was initially observed in numerical experiments.) This is seen in Figures 4 and 6.*

Case 3. (Circular segments) *If* $a = 0,\ b \neq 0$ *meaning that* $v = bc_i = const$ *but* $u = 2bt$ *varies, both* $x(t)$ *and* $y(t)$ *are non-periodic. The curves are circular segments (this will be shown later) between the points* $(0, -1)$ *and* $(0, 1)$ *excluding the points themselves, which are never reached by the curves. These circular segments shown in Figure 5 (in red) are gradient lines to the closed circles of the Case 1 .*

We also note that the red curves in Figure 5 represent spirals degenerated into circular segments bounded by the foci yet *never reaching* them. This is as the theory suggests for polynomial systems.

Now we determine the points where the trajectory crosses the y axis in Cases 1, 2.

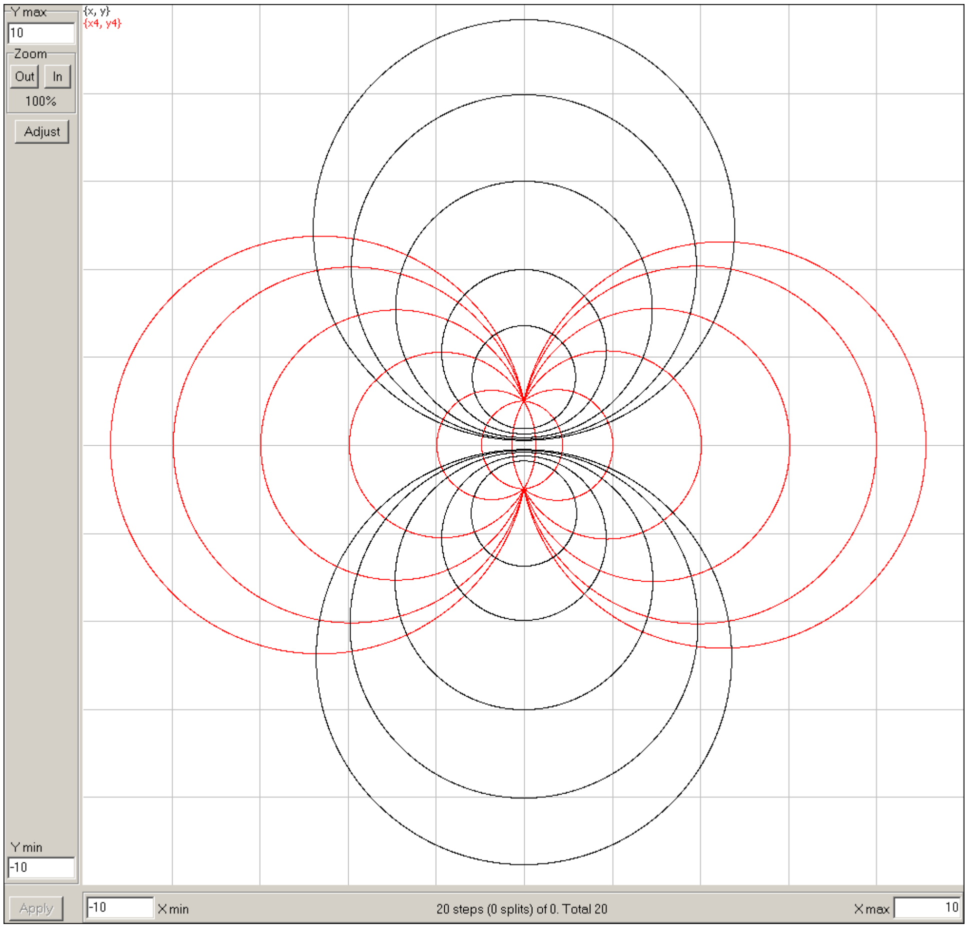

FIGURE 5. CirclePortrait.

THEOREM 1. *For $a \neq 0$ (Cases 1 and 2) the infinite sequence of points where the circle, or the single or double spiral crosses the y axis is defined by*

$$y_k = \begin{cases} \dfrac{e^{u_k}+1}{e^{u_k}-1}, & k = 2n \\ \\ \dfrac{e^{u_k}-1}{e^{u_k}+1}, & k = 2n+1 \end{cases}, \tag{3.6}$$

where[2]

$$u_k = \frac{2(a^2+b^2)c_i - 2abc_r - kb\pi}{a}, \qquad k \in \mathbb{Z}.$$

[2] Generally $y_{2n} \neq \dfrac{1}{y_{2n+1}}$.

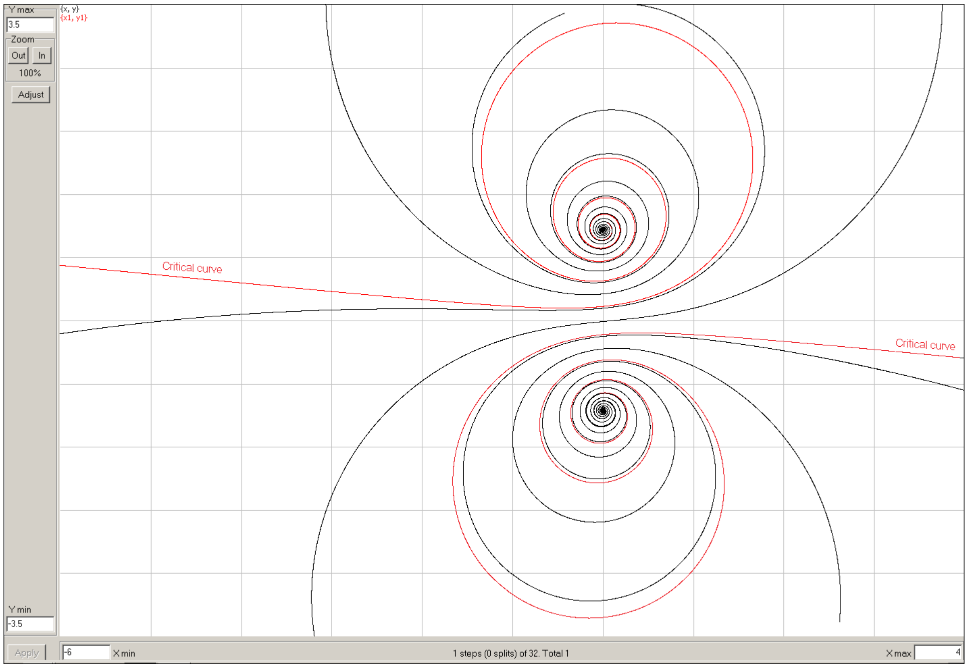

FIGURE 6. Spiral Portrait with Asymptote.

For $b = 0$ *the sequence represents the diametrical points of the circle. Otherwise for* $b \neq 0$ *this sequence represents spirals: typically one double spiral, unless the critical value*

$$k_0 = 2\frac{(a^2 + b^2)c_i - abc_r}{b\pi} \tag{3.7}$$

happens to be integer and even. *Then the sequence represents two disconnected single spirals having an asymptote. If* k_0 *is an integer but* odd, *the double spiral passes through the origin and is comprised of two congruent pieces. Otherwise, the two pieces of the double spiral are incongruent and the spiral does not pass through the origin.*

PROOF. Let $a \neq 0$ and v not be a constant in Equation 3.5. With this in mind, choose the points v_k for which $\sin v = 0$, namely

$$v_k = k\pi, \qquad k \in \mathbb{Z}.$$

The respective $t_k,\ u_k$, according to formula (3.5) are

$$\begin{aligned} v_k &= -2at - 2ac_r + 2bc_i = k\pi \\ t_k &= \frac{2bc_i - 2ac_r - k\pi}{2a} \\ u_k &= -\frac{b}{a}k\pi - 2bc_r + \frac{2(a^2+b^2)c_i}{a}, \qquad k \in \mathbb{Z}. \end{aligned} \tag{3.8}$$

Observe that if $b = 0$, all u_k are the same constant (this is the case of circles, see Figure 5). Further on we consider the case $b \neq 0$.

Depending on the sign of $\frac{b}{a}$, u_k monotonically increases or decreases between $-\infty$ and ∞, changing sign at a certain value k depending on the parameters c_r, c_i, a, b. Correspondingly, when u_k changes its sign, e^{u_k} switches from being > 1 or < 1 respectively. Therefore, the values u_k comprise a sequence of (generally) *non-integer* equidistant values with the period $\frac{\pi b}{a}$ of which only one u_k may happen to pass through 0 for the integer value $k = k_0$ (given in Equation 3.7) obtainable if expression (3.8) is equated to zero

$$-\frac{b}{a}k\pi - 2bc_r + \frac{2(a^2+b^2)c_i}{a} = 0$$

and for a given combination of all the parameters c_r, c_i, a, b fraction (3.7) is an integer. For example, Equation (3.7) yields an integer if for some integer n the value c_r is chosen such that $c_r = \frac{2(a^2+b^2)c_i - nb\pi}{2ab}$.

For every point where $\sin v_k = 0$, $\cos v_k = \pm 1$. More specifically, for $k = 2n$, $\cos v_k = 1$, and for $k = 2n+1$, $\cos v_k = -1$. First assume that k_0 is not integer so that for no u_k the denominator $(e^{u_k} \pm 1)^2$ reaches zero and the formula (3.4) yields the following sequences.

When $\cos v_k = -1$

$$y_k = \frac{e^{2u_k} - 1}{e^{2u_k} + 2e^{u_k} + 1} = \frac{(e^{u_k} - 1)(e^{u_k} + 1)}{(e^{u_k} + 1)^2} = \frac{e^{u_k} - 1}{e^{u_k} + 1}, \quad k = 2n+1$$

and when $\cos v_k = 1$

$$y_k = \frac{e^{2u_k} - 1}{e^{2u_k} - 2e^{u_k} + 1} = \frac{(e^{u_k} - 1)(e^{u_k} + 1)}{(e^{u_k} - 1)^2} = \frac{e^{u_k} + 1}{e^{u_k} - 1}, \quad k = 2n.$$

Here $|y_{2n+1}| < 1 < |y_{2n}|$ so that all odd $y_{2n+1} \in (-1,\ 1)$. For the positive part of the spiral (above the x axis) the odd y coordinates y_{2n+1} correspond to the lowest point in laps of the spiral, while the even y coordinates y_{2n} correspond to the top points in the laps (vice versa for the negative part). Since the values u_k are equidistant and by assumption neither is zero, there are two nearest to 1, the even points u_{2n}, u_{2n+2} (1 being in between, but not in the middle as by the assumption $u_{2n+1} \neq 0$ - see Figure 7).

Therefore, the two largest laps of the spiral (positive and negative) correspond to these two points, and the larger one of these corresponds to that of the two which is closer to 1. Thus, with non-integer k_0 the parts of the spiral are incongruent.

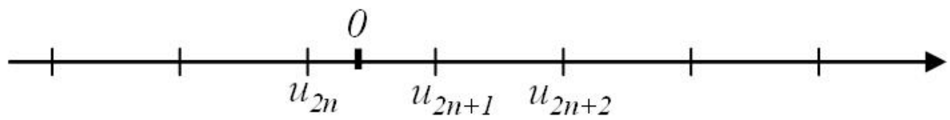

FIGURE 7. The equidistant grid of ...$u_{2n},\ u_{2n+1},\ u_{2n+2}$... with 0 between u_{2n} and u_{2n+2} yet $u_{2n+1} \neq 0$.

If k_0 is an integer and *odd* while 0 is between $u_{2n},\ u_{2n+2}$, then $u_{2n+1} = 0$ so that the parts below the x axis and above it are congruent. Note that if $a \neq 0,\ b \neq 0$, and k_0 is not critical then all spirals are double spirals. Points A and B of the x axis are mapped by the respective spirals onto an infinite sequence of points onto the y axis. This is seen in Figure 4.

Finally if the critical value k_0 is an integer and *even* (see Figure 6), we have to establish the existence of particular limits

$$\lim_{u \to u_{k_0} = \pm 0} x(t) = \mp\infty$$

$$\lim_{u \to u_{k_0} = \pm 0} y(t) = \pm\infty$$

$$\lim_{u \to u_{k_0} = 0} \frac{y(t)}{x(t)} = -\frac{b}{a}$$

(we assumed that $a \neq 0$ and $b \neq 0$). Observe from Equation (3.5) that

$$v(t) = \frac{2(a^2 + b^2)c_i - 2abc_r - au}{b}$$

and from Equation (3.7) that

$$\frac{2(a^2 + b^2)c_i - 2abc_r}{b} = k_0\pi = 2n\pi$$

so that for the critical k_0 $v = 2n\pi - \dfrac{au}{b}$.

Denoting $r = \dfrac{a}{b} \neq 0$, we have

$$\begin{aligned} \sin v &= -\sin ru \\ \cos v &= \cos ru. \end{aligned}$$

This allows us to obtain the limits as $u \to 0$ in Equation (3.4). Recall that the denominator in fraction (3.4) is non-negative so that the sign of both fractions is determined by the numerator only. Observe that with $u \to 0$

$$e^{2u} - 2e^u \cos ru + 1 =$$

$$= 2 + 2u + \frac{4u^2}{2!} + \frac{8u^3}{3!} + ... - 2(1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + ...)(1 - \frac{r^2u^2}{2!} + \frac{r^4u^4}{4!} + ...) =$$

$$= 2 + 2u + \frac{4u^2}{2!} + \frac{8u^3}{3!} + ...$$

$$-2\left(1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + ... - \frac{r^2u^2}{2!} - \frac{r^2u^3}{2!} - \frac{r^2u^4}{2!2!} - \frac{r^2u^5}{2!3!} + ... + \frac{r^4u^4}{4!} + \frac{r^4u^5}{4!} + ...\right) =$$

$$= \frac{4u^2}{2!} + \frac{8u^3}{3!} + ... - 2\left(\frac{1-r^2}{2!}u^2 + \left(\frac{1}{3!} - \frac{r^2}{2!}\right)u^3 + ...\right) =$$

$$= \frac{2-2r^2}{2!}u^2 + \left(\frac{9}{3!} - \frac{r^2}{2!}\right)u^3 + ...$$

so that the lowest order of u is 2 or higher. At that the numerator

$$\begin{aligned} 2e^u \sin v &= -2e^u \sin ru = -2(1 + u + \frac{u^2}{2!}...)(ru - \frac{r^3u^3}{3!} + ...) = \\ &= -2(ru - \frac{r^3u^3}{3!} + ... + ru^2 - \frac{r^3u^4}{3!} + ...) = \\ &= -2ru - 2ru^2 + \frac{2r^3u^3}{3!} + ... \end{aligned}$$

is of order 1, and the numerator $e^{2u} - 1$ is of order 1 in u, also. Therefore, with $u \to 0$ both fractions in Equation (3.4) may be reduced by u leaving us with nonzero numerators and the denominator approaching 0. This proves the first two limits.

Finally, with $u \to 0$

$$\lim_{u\to 0} \frac{y(u)}{x(u)} = \lim_{u\to 0} \frac{e^{2u}-1}{2e^u \sin v} = \lim_{u\to 0} \frac{e^{2u}-1}{2e^u \sin v} =$$

$$= \lim_{u\to 0} \frac{2u + \frac{4u^2}{2!} + \frac{8u^3}{3!}...}{-2ru - 2ru^2 + \frac{2r^3u^3}{3!} + ...} = \lim_{u\to 0} \frac{2 + \frac{4u}{2!} + ...}{-2r - 2ru + \frac{2r^3u^2}{3!} + ...} = -\frac{1}{r} = -\frac{b}{a}$$

meaning that the asymptote does exist. □

In Figure 6 we display Case 2 when $a \neq 0,\ b \neq 0$, and the value of k_0 is critical. One sees that some spirals have degenerated into a single spiral (above and below the red critical curve) which do not cross the x axis. Those in between the red curves are double spirals and they do cross the x axis.

We have the following corollary

Corollary 1. *($b = 0$, $u = const$) The four points above the x axis corresponding to $0°,\ 90°,\ 180°,\ 270°$, the center, and the radius are, respectively*

$$\left(\frac{2e^u}{e^{2u}-1}, 0\right),\ \left(0, \frac{e^u+1}{e^u-1}\right),\ \left(-\frac{2e^u}{e^{2u}-1}, 0\right),\ \left(0, \frac{e^u-1}{e^u+1}\right),\ \left(0, \frac{e^{2u}+1}{e^{2u}-1}\right),\ \frac{2e^u}{e^{2u}-1}$$

where the 90° *and* 270° *locations correspond to* $\cos v = \pm 1$ *in Equation (3.4). Only in this case does* $y_{2n} = \dfrac{1}{y_{2n+1}}$ *in formula (3.6).*

REMARK 2. *Though the points defined by* 90° *and* 270° *correspond to* $\cos v = \pm 1$, *the locations* 0° *and* 180° *on the circle (quite counter-intuitively)* do not *correspond to* v *where* $\cos v = 0$ *and* $\sin v = \pm 1$ *in Equation (3.4). The proof is given below.*

PROOF. As it follows from Equations (3.4) and (3.6), the points corresponding to 90° and 270° on the y axis correspond to $x = 0$, or $\sin v = 0$, so that $\cos v = \pm 1$ in Equation (3.4). Therefore,

$$y_{90^\circ} = \frac{e^{u_k}+1}{e^{u_k}-1}, \quad y_{270^\circ} = \frac{e^{u_k}-1}{e^{u_k}+1},$$

the center is

$$\frac{1}{2}(y_{90^\circ} + y_{270^\circ}) = \frac{1}{2}\left(\frac{e^u-1}{e^u+1} + \frac{e^u+1}{e^u-1}\right) = \frac{e^{2u}+1}{e^{2u}-1}$$

and the radius is

$$R = y_{90^\circ} - \frac{e^{2u}+1}{e^{2u}-1} = \frac{e^u+1}{e^u-1} - \frac{e^{2u}+1}{e^{2u}-1} = \frac{2e^u}{e^{2u}-1}.$$

Observe that at the points v where $\cos v = 0$ the formula (3.4) yields the value $x = \dfrac{2e^u}{e^{2u}+1}$ rather than $R = \dfrac{2e^u}{e^{2u}-1}$ as noted in Remark 2. The extreme points of $x = x(v)$ *are not* at the v where $\cos v = 0$. We will now determine at which v it does occur.

With $u = const$, value $x = x(v)$ reaches its maximum and minimum where the derivative x'_v changes its sign

$$x'_v = \frac{(2e^u \cos v)\,(e^{2u} - 2e^u \cos v + 1) - (2e^u \sin v)\,(2e^u \sin v)}{e^{2u} - 2e^u \cos v + 1} = 0,$$

$$e^{2u}\cos v + \cos v - 2e^{2u} = 0$$

so that the maximum and minimum of $x = x(v)$ are reached when

$$\cos v = \frac{2e^u}{e^{2u}+1} \text{ and } \sin v = \frac{e^{2u}-1}{e^{2u}+1}.$$

Substituting these values into the Equation (3.4) for x we can see that

$$\frac{2e^u \sin v}{e^{2u} - 2e^u \cos v + 1} = \frac{2e^u \dfrac{e^{2u}-1}{e^{2u}+1}}{e^{2u} - 2e^u \dfrac{2e^u}{e^{2u}+1} + 1} = \frac{2e^u(e^{2u}-1)}{e^{4u} + e^{2u} - 4e^{2u} + e^{2u} + 1} = \frac{2e^u}{e^{2u}-1}$$

and, thus

$$x_{180^\circ,0^\circ} = \pm\frac{2e^u}{e^{2u}-1} = \pm R.$$

Finally, in the formula (3.6) for circles $y_{2n} = \dfrac{1}{y_{2n+1}}$ because the terms in the sequence y_k take only two values. □

COROLLARY 2. *If $a = 0$, and $v = const$ then the point of intersection with the x axis, the center, and the radius of circular segments for the trajectories (Case 3, Figure 5) are, respectively*

$$\left(\frac{1-\cos v}{\sin v}, 0\right), \; -\frac{\cos v}{\sin v}, \; \frac{1}{\sin v}.$$

PROOF. From Equation (3.4)

$$y = \frac{e^{2u} - 1}{e^{2u} - 2e^u \cos v + 1} = 0$$

only when $u = 0$ so that the point of intersection with the x axis is given by

$$x_0 = \frac{2e^u \sin v}{e^{2u} - 2e^u \cos v + 1} = \frac{2 \sin v}{1 - 2\cos v + 1} = \frac{\sin v}{\cos v + 1} = \frac{1-\cos v}{\sin v}.$$

(The sign of x_0 depends on the sign of $\sin v$). If c_0 denotes the center, and R - the radius of the circle, then $R = |x_0 - c_0|$, and $R^2 = c_0^2 + 1$, so that

$$c_0 = \frac{x_0^2 - 1}{2x_0} = \frac{\left(\dfrac{1-\cos v}{\sin v}\right)^2 - 1}{2\dfrac{1-\cos v}{\sin v}} = -\frac{\cos v}{\sin v}$$

and $R = \dfrac{1}{\sin v}$. □

COROLLARY 3. *For spirals, the parameter b in $u(t)$ in Equation (3.5) and in e^u in Equaiton (3.4) is a measure of steepness of the spiral, or how steeply its laps wind and tighten around the focus (never reaching it). Thus, $b = 0$ (Case 1) corresponds to circles and zero steepness. If b is near zero, the value e^u changes slowly within one period so that spiral laps resemble a circle. On the contrary, if $a = 0$ $(b \neq 0)$ the spirals straighten achieving vertical steepness. They no longer wind around the focus (Case 3). Instead, the curves degenerate into circular segments bounded by the foci yet* never reaching *them - just like spirals never do. Due to the polynomial form of the ODEs, all finite points of the phase space are regular; the uniqueness of the solution holds at each of these points and the trajectories never cross each other. The solutions corresponding to the two foci $x \equiv 0$, $y \equiv \pm 1$ (Figure 5) are stationary.*

COROLLARY 4. *Every curve satisfying Equation (3.4) of the double spiral crosses the x axis at one and only one point where $u(t) = 0$ meaning the numerator $e^{2u} - 1 = 0$ in the formula for y. Therefore, every curve of the phase portrait of the respective ODE (3.2) is uniquely defined by its crossing point with the x axis. As the ODE (3.2) is polynomial, no finite point of its phase space is singular, the curves of the phase portrait cannot cross each other, and therefore the mapping* *(the points of the x axis) $\leftrightarrow$ (the curves of the phase portrait)*

is bijective as can be seen in Figures 5, 4 , 6.

COROLLARY 5. *Each double spiral crosses the y axis at an infinite number of places which comprise two sequences that converge to the two respective centers. The distances between the corresponding points in each sequence approach 0 as Figure 4 shows.*

COROLLARY 6. *Due to the bijective mapping*

(the double spirals) $\leftrightarrow$ (the points of the x axis),

(a) Every double spiral maps the respective point of the x axis onto the y axis infinitely many times; (b) The entire ray of the x axis with the vertex at the given point maps infinitely many times onto the more and more narrow gaps on the y axis between the respective laps of the spiral. One observes this in Figure 4.

We now go onto Case 3.

THEOREM 2. *For $a \neq 0,\ b = 0$ (Case 1) or $a = 0,\ b \neq 0$ (Case 3) the curves are circles or circular segments such that the family of curves of the Case 3 are the respective gradients (perpendiculars) to the curves of the family in the Case 1 (see Figure 5).*

PROOF. For $a \neq 0,\ b = 0$ the center and the radius of the supposed circle are respectively $c_0 = \dfrac{e^{2u}+1}{e^{2u}-1}$ and $R = \dfrac{2e^u}{e^{2u}-1}$. A straightforward calculation shows formula (3.4) satisfies the equation

$$x^2 + (y - c_0)^2 = R^2.$$

For $a = 0,\ b \neq 0$ the center and the radius of the supposed circle are respectively $c_0 = \dfrac{\cos v}{\sin v}$ and $R = \dfrac{1}{\sin v}$. Again, a straightforward calculation shows that the parametric form (3.4) satisfies the equation

$$(x - c_0)^2 + y^2 = R^2.$$

The family of curves for $a \neq 0,\ b = 0$ satisfy the ODEs (3.3) and have the form

$$\begin{aligned} x_1' &= a(x_1^2 - y_1^2 + 1) \\ y_1' &= 2ax_1y_1. \end{aligned}$$

The family of curves for $a = 0,\ b \neq 0$ satisfy the ODEs (3.3) and have the form

$$\begin{aligned} x_2' &= -2bx_2y_2 \\ y_2' &= b(x_2^2 - y_2^2 + 1). \end{aligned}$$

The scalar product at an arbitrary point $(x,\ y)$ is given by

$$((x_1',\ y_1'), (x_2',\ y_2'))|_{(x,\ y)} = (x_1'x_2' + y_1'y_2')\,|_{(x,\ y)}$$
$$= -2ab(x^2 - y^2 + 1)xy + 2ab(x^2 - y^2 + 1)xy = 0.$$

$\square$

4. Conclusions

This research is an example of a mathematical study triggered by observation of numerical experiments at a computer, followed by the conventional mathematical analysis of the observed phenomenon. Not only did this mathematical analysis succeed in establishing the proof and explanation for what was observed, but it also revealed existence of the critical asymptotic curve (Figure 6). This curve would have been impossible to find by numerical experiment only. The numerical experimenting and the analytic approaches complemented each other beautifully in discovering an amazing property of a particular planar polynomial system of ODEs.

In Appendix 1 we discuss some of the history of this problem. In Appendix 2 we explain to the reader how to obtain and use Taylor Center.

5. Appendix 1: Brief History

As was mentioned in this article, this research was prompted by an example of a class of planar polynomial IVP ODEs. As outlined in Gofen [**2**]; Not just planar, but a large class of ODEs whose right hand sides are general elementary functions can be made rational, polynomial and then quadratic. Hilbert's 16^{th} problem is about polynomial planar ODEs. In fact, polynomial ODEs are more encompassing than Hilbert knew. Sochacki [**7**] outlines the properties of polynomial ODEs.

It was an interest in why quadratic ODEs are ubiquitous that led to the study of ODEs of the form 1.1. It was through numerical experiments that the double spirals of

$$\begin{aligned} x' &= x^2 - y^2 + 2xy - x - 3.5y + 1, \qquad x(0) = 0.1 \\ y' &= -x^2 + y^2 + 2xy - y + 3.5x - 1, \qquad y(0) = 0 \end{aligned}$$

were discovered (Figure 1) using Taylor Center. It was then found through numerical experiment that the size of the spirals did not have a direct relationship to the initial conditions.

Upon studying more classes of the ODEs Lucas and Sochacki [**6**] discovered some periodic solutions and periodic properties of these periodic trajectories. In this article, we presented the theory of more properties of the ODE 1.1 discovered using Taylor Center. The Taylor Center is an advanced ODE solver and curve plotter and was indispensable in conducting the research presented here. Indeed, dynamic drawing of trajectories shows and reveals much more than a still image does.

6. Appendix 2: Taylor Center Software

The graphs and ODEs referred in this article can be replayed in a free demo-version of an advanced ODE solver for PCs under Windows implementing the modern Automatic Differentiation Taylor method for integration of ODEs. One of many features of this software is that it displays and dynamically plays the motion along the trajectories in the real time so that the users can observe the effects reported in this article on their own.

Follow the instructions on the website [**4**] for the installation of this software which comes with the folder of Samples of various problems, and which contains also the special sub-folder *DoubleSpiralStudy* of the samples related to this article. For each ODE and figure presented in this article there is a script file name which can be navigated. They can be opened in the menu item *File/Open script*.

When such a script file is successfully opened, it draws the respective trajectory solution. You may wish to maximize the graph window to see the trajectory in more details. In order to Play the trajectory in real time, press the Play button. For the next example, return to the main window and repeat *File/Open script* menu item. You can watch other illustrative examples (unrelated to the article) either in the menu item Demo, or by opening other script files in the folder Samples. (For viewing 3D stereo examples you will need a pair of Red/Blue glasses).

Below we describe how the reader is able to reproduce the examples discussed in this article, and to actually watch the dynamic behavior reported there.

When you click *Play* and watch the real time motion along the trajectory of the solution, you will notice that it takes the same time to run around every lap of the spiral whatever the diameter of the lap. It was this remarkable periodicity which increased our interest in these ODEs.

Using this very software, we accurately computed these time intervals taken by every lap. We expected to figure out that the intervals are only approximately equal, yet it appeared that they were equal exactly (with 18 decimal digit accuracy). This finding begged for further research. Below is the description of how to follow the research.

Under *File/Open Script* load the example *CircleODEs.scr* observing that the circular laps take the same time to trace independent of the diameter. *Verification.scr* demonstrates a double spiral with general type parameters ($a = 0.5,\ b = 0.05$) drawn both parametric and as an ODE solution (only the red curve is visible because of exact congruence of both solutions). The *SteepSpiral.scr* represents a case of a very steep spiral ($a = b = 1$) in which only one big lap is visible while the laps winding around the foci are too small to display.

VerificationAsympt.scr and *VerificationAsympt1.scr* demonstrate the case when the double spiral broke into two asymptotic pieces.

CircSegment.scr presents the Case 3 of a circular segments when $a = 0, b = 1$.

With a two monitor system, it is recommended to place the Graph window onto a separate monitor and keep it maximized for higher resolution of curve drawing. The main panel (having tabbed pages) allows to inspect the source ODEs, Auxiliary equations, Initial values, and Constants on the page Equation setting. (See the Help or the User Manual for more explanations).

References

[1] Brown,J.,Churchill,R.,*Complex Variables and Applications*,9^{th} edition,McGraw Hill, 2013.

[2] A. Gofen, The ordinary differential equations and automatic differentiation unified. *Complex Variables and Elliptic Equations*, **54**, 2009, 825–854.

[3] A. Gofen, The Unifying View on ODEs and AD yet with a gap to fill, *9th AIMS Conference on Dynamical Systems, Differential Equations and Applications*, Orlando, July 2012.

[4] A. Gofen, *The Taylor Center and its description Web page*, http://www.ski.org/gofen/TaylorMethod.htm.

[5] Hirsch, M.,Smale, S.,Devaney,R.,*Differential Equations, Dynamical Systems, and an Introduction to Chaos*,3^{rd} edition,Academic Press, 2013.

[6] S. Lucas, J. Sochacki, Simple heteroclinic orbit examples in the place, *9th AIMS Conference on Dynamical Systems, Differential Equations and Applications*, Orlando, July 2012.

[7] J. Sochacki, Differential Equations - Examples, Solutions, Properties. *Neural Parallel & Scientific Computations*, **18**, 2010, 441–450.